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Stratified pooling versus full pooling: (non-) emptiness of the core

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Abstract

We study a situation with several service providers that are located geographically close together. These service providers keep spare parts in stock to protect for downtime of their high-tech machines and face different downtime costs per stock-out. Service providers can cooperate by forming a joint spare parts pool, and we study the allocation of the joint costs to the individual service providers by studying an associated cooperative game. In the extant literature, the joint spare parts pool is typically controlled by a suboptimal full-pooling policy. This may lead to an empty core of the associated cooperative game. We show possible emptiness of the core under a full-pooling policy in our setting as well.

The focus of the paper is then on situations in which we allow service providers to apply an optimal policy: a kind of stratification that determines, depending on the real-time on-hand inventory, which service providers may take parts from the pool. We formulate the associated cooperative game, which we call a stratified pooling game, by defining each coalitional value in terms of the minimal long-run average costs of a Markov decision process. We show that the core of stratified pooling games is always non-empty. Our five-step proof of this result is of interest in itself because it may be more generally applicable for other cooperative games where coalitional values correspond to the minimal long-run average costs of Markov decision processes.

1 Introduction

In the last decades, spare parts pooling has shown its potential in several industries, including the airline industry (Kilpi and Vepsäläinen [8]) and the electricity market

(Kukreja et al. [11]). In these industries, one typically applies a full-pooling policy, which means that a joint spare parts pool is formed (which may, e.g., consist of the original stock points of the participating players or one (new) joint stock point) from which *every* participating player can demand as long as the joint spare parts pool is non-empty. Despite the fact that this form of pooling may reduce long-run average costs significantly, it is not per se *optimal*. For instance, when the criticality of a specific spare part may differ per party, full pooling may be *far* from optimal (see, e.g., Koçağa and Şen [9], Kranenburg and van Houtum [10], and Wieczorek et al. [21]). As an example, one can think of two service providers that each keep the same spare parts in stock for the same type of machines, but face different downtime costs for their machines (e.g., when the service providers have different contractual agreements with their customers). When the service providers then decide to pool their spare parts, it may be better to reserve some of the spare parts for the service provider with the higher downtime costs. In this work, we will also consider such an environment, in which service providers can pool their spare parts, face different downtime costs, and apply a pooling policy that is better than full pooling. More specifically, we let the service providers apply an *optimal* pooling policy, which is, in general, not full pooling. Then, we address the main question of this work, namely: how should we allocate the joint total costs of the optimal pooled system amongst the involved parties? So far, such cost allocation problems have received attention in literature for situations in which spare parts are pooled according to a possibly suboptimal pooling policy (see, e.g., Karsten et al. [6] and Schlicher et al. [17]). However, it is unexposed in settings in which spare parts are pooled according to an optimal pooling policy. To the best of our knowledge, we are the first who deal with this issue explicitly.

In this paper, we will address this cost allocation problem as follows. First, we describe the underlying spare parts situation, which consists of several service providers that are located geographically close together. These service providers stock the same repairable spare parts to protect for downtime of their high-tech machines and send a failed part of a high-tech machine to their own repair shop, which repairs such parts one-by-one. The frequency by which the high-tech machines fail may vary per service provider. If no spare part is available upon demand, an external spare part is leased from an external supplier with infinite supply for the duration of the repair time of the failed component. The expected costs associated with such a situation (like the extra downtime costs of the high-tech machine, the transhipment costs of the leased spare part, and so on), are called downtime costs and may vary amongst the service providers as well.

As a second step, we formulate an associated cooperative game. For this game, we assume that players (i.e., the service providers) cooperate by forming a joint spare parts pool in which all failed parts of all high-tech machines are sent to the same repair shop, which repairs these parts one-by-one. Operating the joint pool involves, for each spare parts demand, the decision problem of satisfying or rejecting demand. Whenever a demand is rejected (and so the spare part is not satisfied from the joint spare parts pool), a similar emergency procedure is instigated as discussed before, with downtime costs that depend on whose demand is rejected. We show that, under a full-pooling policy that satisfies all demand while spare parts are available, the core of the associated game can be empty. The focus of the paper is then on situations in which an optimal pooling policy is applied, i.e., a pooling policy that minimizes the joint long-run average downtime costs per time unit. It turns out, based on a classical result of Ha [5], that such an optimal pooling policy has the form of a critical level policy. Such critical level policies are characterised as follows. According to the different downtime costs, demand for a spare part of a player is accepted if and only if the total inventory level of the joint spare parts pool is above a certain critical level. In this way, some spare parts are reserved for the players with relatively high downtime costs. So, there is some kind of stratification that determines when players are allowed to make use of the joint spare parts pool. For that reason, we refer to this form of pooling as *stratified pooling* and to the associated cooperative game as a stratified pooling game.

We investigate stratified pooling games on core non-emptiness, i.e., we investigate whether there always exists an allocation of the joint costs such that no group of players has a reason to break up from the collaboration. For proving core non-emptiness, we face the problem that the optimal pooling policy may differ per coalition. To this end, we set up a 5-step proof, which uses that the underlying spare parts situation can be described by a Markov decision process (MDP) and the optimal pooling policy as a stationary policy in this MDP. With these new modelling and proof techniques, we are able to show that the core of stratified pooling games is always non-empty. We want to emphasize that these techniques may be more generally applicable for operations research games where coalitional values correspond to the minimal long-run average costs of MDPs as well.

Although stratified pooling games originate from spare parts situations, they are not limited to this specific type of situations only. For instance, in line with the work of

Ha [5], the service providers could represent single-server produce-to-stock production facilities that each face different penalty costs for their lost sales. Then, these production facilities can collaborate by pooling their inventories, customer streams, and production rates into a joint system where new products are produced by a single server.

Stratified pooling games fit within the class of operations research games (see, e.g., Borm et al. [4]) and in more detail within the class of cooperative resource pooling games. These games have in common that resources (e.g., spare parts, hospital beds, repair men, or machines) are pooled amongst the players. In the last couple of years, several of these games have been investigated and we will shortly discuss them below. Here, we restrict ourselves to resource pooling games in which queueing models are incorporated. We categorize these resource pooling games into two categories, namely games in which solely queueing systems are pooled and games in which, in addition to pooling of queueing systems, spare parts are pooled as well.

In the first category, in which solely queueing systems are pooled, we find the work of Anily and Haviv [1], Karsten et al. [7], Timmer and Scheinhardt [20], and Anily and Haviv [2]. Anily and Haviv [1] study a single-server queueing game, in which each player has an own server with a fixed service rate. The players can pool these service rates and their individual exogenous driven customer streams into a single $M/M/1$ queueing system, whose service rate is the sum of the service rates of the players that collaborate. In doing so, the players can reduce waiting time of customers. Anily and Haviv [1] show that the core of this game is always non-empty and give an explicit expression for all nonnegative core allocations of this game. Karsten et al. [7] study a variant of this game, by assuming that the collaboration is modelled as an $M/M/s$ queueing system instead of an $M/M/1$ queueing system. For this specific game, in which players face a fixed cost rate per server and homogeneous costs for waiting customers, Karsten et al. [7] provide a sufficient condition for core non-emptiness as the core can be empty in general. Timmer and Scheinhardt [20] study cooperative games associated with general Jackson networks. Each player owns a single-server station, which is modelled as an $M/M/1$ queue. The players can cooperate by redistributing their combined service capacities amongst the stations to reduce total waiting time. They show that the core is non-empty in general. Anily and Haviv [2] study parallel $M/M/1$ queues. They consider both waiting and no waiting in the queue. In doing so, they consider cooperation via capacity sharing as in Timmer and Scheinhardt [20] as well as cooperation under unobservable routing. The latter form of cooperation boils down to an optimal division of arrival rates over the stations. In all cases, core non-emptiness is proven.

In the second category, in which, in addition to pooling of queueing systems, spare parts are pooled as well, we find the work of Karsten et al. [6] and Schlicher et al. [17]. Karsten et al. [6] study a setting with several players who stock expensive, low-demand, repairable spare parts for their high-tech machines. These players can collaborate by full pooling of their spare parts via free transshipments. The authors model this pooled spare parts situation as an $M/M/s/s$ queueing system (better known as an Erlang loss system), in which (in terms of standard queueing terminology) a customer can be seen as a demand for a spare part and a server as a spare part in repair. In contrast to the queueing systems of Anily and Haviv [1] and Karsten et al. [7], in the $M/M/s/s$ queueing system, customers are not allowed to wait. Karsten et al. [6] show that the core of the associated game can be empty and subsequently provide a sufficient condition for core non-emptiness. Schlicher et al. [17] study a variant of the model of Karsten et al. [6] for a restricted domain: they assume that each player keeps exactly one spare part in stock and assume that each player has the same demand rate. For this setting, Schlicher et al. [17] let the players pool their spare parts via a fixed suboptimal critical level policy instead of a full-pooling policy. This pooled spare parts situation is also modelled as a queueing system, in which the demand rate of the customers is state dependent. Schlicher et al. [17] show that, for the restricted domain, the core is always non-empty.

Our paper can be categorized in the second category as well: we allow the players to pool their repairable spare parts and model the pooled repair process as an $M/M/1$ queueing system. With respect to the type of spare parts pooling, we deviate from Karsten et al. [6] and Schlicher et al. [17] by applying an optimal spare parts pooling form. With respect to the form of queuing system under pooling, we follow Anily and Haviv [1] and so, it is possible that a failed part has to wait before repair can actually start. Our main result is that, under these modelling assumptions, the core is always non-empty. This result is of particular interest, because the forms of pooling that have been investigated in literature (e.g., full pooling) cannot in general guarantee a non-empty core.

In Table 1 an overview of the most relevant resource pooling games is represented, according to the various modelling assumptions and whether the core (of the related resource pooling game) is non-empty in general or not.

Table 1: Classification of the most relevant cooperative resource pooling games according to various modelling assumptions as well as the result on core non-emptiness

		Queuing system form under pooling	
		Single	Multiple
Spare parts pooling policy	Full pooling	**This paper** (*waiting in queue, core can be empty*)	Karsten et al. [6] (*no waiting in queue, core can be empty*)
	Critical level pooling	—	Schlicher et al. [17] (*no waiting in queue, core is non-empty*⋆)
	Optimal pooling	**This paper** (*waiting in queue, core is non-empty*)	—
	Not applicable (no spare parts)	Anily and Haviv [1] (*waiting in queue, core is non-empty*)	Karsten et al. [7] (*waiting in queue, core can be empty*) Timmer and Scheinhardt [20] (*waiting in queue, core is non-empty*) Anily and Haviv [2] (*waiting and no waiting in queue, core is non-empty*)

⋆ This result holds for a restricted domain only.

Now, we summarize the main contributions of this paper:

- We are the first who analyze a resource pooling game in which spare parts are pooled in an optimal way (instead of in a possibly suboptimal way, like, e.g., full pooling).
- We prove that the core of stratified pooling games is always non-empty. This result is of particular interest as for several other spare parts pooling games (in which a suboptimal pooling form is assumed) the core may be empty. In particular, if players would naively apply a full-pooling policy in our stratified pooling game, core non-emptiness is also not guaranteed.
- We present a 5-step proof for core non-emptiness of stratified pooling games, which is of interest in itself. To the best of our knowledge, we are the first who relate a cooperative game to an MDP and so succeed to prove that the core is always non-empty. These modelling and proof techniques may be more generally applicable for operations research games where coalitional values correspond to the minimal long-run average costs of MDPs as well.

The remainder of this paper is organised as follows. We start in Section 2 with preliminaries on cooperative game theory as well as on MDPs. Subsequently, we introduce spare parts situations and the associated stratified pooling games in Section 3. In Section 4, we will show our main result that stratified pooling games have a non-empty core. In Section 5 conclusions are drawn. Proofs of lemmas are relegated to the appendix.

2 Preliminaries

In this section, we provide some basic elements of cooperative game theory as well as of (discrete time) Markov decision processes.

2.1 Cooperative Game Theory

Consider a finite set of *players* $N = \{1, 2, \ldots, n\}$ and a function $c : 2^N \to \mathbb{R}$ called a *characteristic function*, with $c(\emptyset) = 0$. The pair (N, c) is called a *cooperative cost game with transferable utility*, shortly called *game*. A subset $S \subseteq N$ is a *coalition* and $c(S)$ represents the costs incurred by the players in S. The costs can be transferred freely among the players. The set N is called the *grand coalition*. A *cost vector* for a game (N, c) is a vector $x \in \mathbb{R}^N$ describing how to allocate the costs, where player $i \in N$ is allocated x_i. A cost vector $x \in \mathbb{R}^N$ is called *efficient* if $\sum_{i \in N} x_i = c(N)$. This implies that all costs are distributed amongst the players of the grand coalition N. A cost vector $x \in \mathbb{R}^N$ is called *stable* if no group of players has an incentive to leave the grand coalition N, i.e., $\sum_{i \in S} x_i \leq c(S)$ for all $S \subseteq N$. The set of efficient and stable cost vectors of (N, c), called the *core* of (N, c), is denoted by $\mathscr{C}(N, c)$.

2.2 Discrete Time Markov Decision Processes

In this section we present some basic concepts of discrete time Markov decision processes (MDPs). An MDP is a mathematical framework for modelling sequential decision problems under uncertainty. Consider a set $T = \mathbb{N} \cup \{0\}$ of time epochs, a countable set $\mathscr{Y}$ of states, a finite set $\mathscr{A}(y)$ of actions for each $y \in \mathscr{Y}$, non-negative costs $C(y, a)$ for each $y \in \mathscr{Y}$ and all $a \in \mathscr{A}(y)$, and transition probabilities $p(y'|y, a)$ for all $y' \in \mathscr{Y}$, all $y \in \mathscr{Y}$, and all $a \in \mathscr{A}(y)$ with $\sum_{y' \in \mathscr{Y}} p(y'|y, a) = 1$ for all $y \in \mathscr{Y}$ and all $a \in \mathscr{A}(y)$. Tuple $(T, \mathscr{Y}, \mathscr{A}, C, p)$ with $\mathscr{A} = (\mathscr{A}(y))_{y \in \mathscr{Y}}$, $C = (C(y, a))_{y \in \mathscr{Y}, a \in \mathscr{A}(y)}$ and $p = (p(y'|y, a))_{y', y \in \mathscr{Y}, a \in \mathscr{A}(y)}$ is called a discrete time Markov decision process.

Let $t \in T$ be a time epoch. A *decision rule* $\omega_t = (\omega_t(y))_{y \in \mathscr{Y}}$ indicates for all states $y \in \mathscr{Y}$ which action to choose at time epoch t. In addition, a *policy* $\omega = (\omega_t)_{t \in T}$ is a sequence of decision rules for all time epochs. Let X_t with $t \in \mathbb{N} \cup \{0\}$ be a random variable that indicates the state at time t. Note that X_t depends on ω and X_0. If initially $X_0 = y \in \mathscr{Y}$, the *long-run average costs per time epoch* under policy ω are

$$J_\omega(y) = \limsup_{n\to\infty} \frac{1}{n} \mathbb{E}_\omega \left[\sum_{t=0}^{n-1} C(X_t, \omega_t(X_t)) | X_0 = y \right].$$

Let Ω be the set of all policies and $J^*(y) = \inf_{\omega\in\Omega} J_\omega(y)$ for all $y \in \mathscr{Y}$. There exists a class of MDPs for which there exists a constant J^* such that $J^* = J^*(y)$ for all $y \in \mathscr{Y}$. In that case, J^* is defined as the *minimal long-run average costs per time epoch* and policy $\omega \in \Omega$ is *optimal* if $J_\omega(y) = J^*$ for all $y \in \mathscr{Y}$. A policy is *stationary* if there exists an f such that $\omega_t = f$ for all $t \in T$. We denote such (stationary) policy by $f = (f(y))_{y\in\mathscr{Y}}$.

For an MDP, the *value function* $V_t(y)$ for all $y \in \mathscr{Y}$ and all $t \in T$ is defined by

$$V_{t+1}(y) = \min_{a\in\mathscr{A}(y)} \left\{ C(y,a) + \sum_{y'\in\mathscr{Y}} p(y'|y,a) \cdot V_t(y') \right\}, \tag{1}$$

with $V_0(y) = 0$ for all $y \in \mathscr{Y}$.

There exists an important result which states that, under two conditions, the minimal long-run average costs per time epoch exist, are attained under a stationary policy and moreover, coincide with the limit of the value function divided by the number of time epochs, when time goes to infinity. In these conditions, one refers to *irreducible* Markov chains and *positive recurrent* Markov chains. A Markov chain, which is attained under a given policy, is said to be *irreducible* if for all states $y \in \mathscr{Y}$ it holds that it can be reached from each other state $y' \in \mathscr{Y}\backslash\{y\}$. A Markov chain is said to be *positive recurrent* if for all states $y \in \mathscr{Y}$ it holds that the expected return time (to state y) is finite. Now, we present this important result, which is derived in Sennott [18, proposition 4.3].

Theorem 1. *Let $(T, \mathscr{Y}, \mathscr{A}, C, p)$ be an MDP. If (i) there exists a stationary policy f inducing an irreducible and positive recurrent Markov chain on $\mathscr{Y}$, and satisfying $J_f(y) < \infty$ for all $y \in \mathscr{Y}$, and (ii) there exists an $\epsilon > 0$ such that $\{y \in \mathscr{Y} | \exists a \in \mathscr{A}(y) : C(y,a) < J_f + \epsilon\}$ is finite, then*

$$J^* = \lim_{t\to\infty} \frac{V_t(y)}{t} \text{ for all } y \in \mathscr{Y}, \tag{2}$$

and moreover, there exists an optimal stationary policy.

In this work, we restrict our attention to MDPs with finite state spaces. This implies that it becomes superfluous to check whether an irreducible Markov chain is positive recurrent as every irreducible Markov chain is positive recurrent in a finite state space (see, e.g., Modica and Poggiolini [13, Theorem 5.71 (ii)]). Moreover, the second condition of Theorem 1 is always satisfied when $\mathscr{Y}$ is finite.

3 Model description

In this section, we introduce spare parts situations and define the associated games, called stratified pooling games. In addition, we discuss that the underlying spare parts situation can be described by a Markov decision process and the optimal pooling policy as a stationary policy in this Markov decision process.

3.1 Spare parts situations

We consider an environment with a finite set $N \subseteq \mathbb{N}$ of service providers that are located geographically close together and each keeps spare parts in stock to prevent costly downtime of their high-tech machines. We limit ourselves to one critical component, i.e., one stock-keeping unit, which is subject to failures. For each service provider $i \in N$, it holds that a failure of a high-tech machine immediately leads to a demand for a spare part. This occurs according to a Poisson process with rate $\lambda_i \in \mathbb{R}_+$. We assume that each service provider $i \in N$ starts with $I_i \in \mathbb{N} \cup \{0\}$ spare part(s) in stock initially. If a spare part is on hand when demand occurs, this demand is always satisfied and the failed part is sent to the repair shop of service provider i, which repairs such parts one-by-one (like in Anily and Haviv [1]). Repair times of these parts are assumed to be independent and identically distributed according to an exponential distribution with mean $\mu_i^{-1} \in \mathbb{R}_+$. If no spare part is available when demand occurs, an emergency procedure is instigated, which means that a spare part is leased (from an external supplier with infinite supply) for the duration of the repair time of the failed component, which is sent to the repair shop (of service provider i). The expected costs associated with the extra idleness of the machine (due to the delivery time of a leased spare part), shipment of an emergency spare part, and so on, shortly called downtime costs, are $d_i \in \mathbb{R}_+$ for service provider i. Finally, we assume that each service provider $i \in N$ is interested in its long-run average costs per time unit. To analyse this setting, we define a spare parts situation as a tuple (N, I, d, λ, μ) with N, $I = (I_i)_{i \in N}, d = (d_i)_{i \in N}$, $\lambda = (\lambda_i)_{i \in N}$ and $\mu = (\mu_i)_{i \in N}$ as defined above. For short, we use θ to refer to a spare parts situation and Θ for the set of spare parts situations.

3.2 Stratified pooling games

Consider spare parts situation $\theta = (N, I, d, \lambda, \mu)$ and coalition $S \subseteq N$ with $S \neq \emptyset$. The players in coalition S can collaborate by pooling their inventories, demand streams, and repair rates (of which the last two are in line with Anily and Haviv [1]) into a joint system with initial inventory level $I_S = \sum_{i \in S} I_i$, (heterogeneous) demand rate $\lambda_S = \sum_{i \in S} \lambda_i$, and a single repair shop, in which components are repaired one-by-one with as repair rate $\mu_S = \sum_{i \in S} \mu_i$. In this joint system, each failed component is sent to the repair shop immediately. Moreover, for each incoming demand, the players face an accept or reject decision problem, which determines whether demand is satisfied from the joint spare parts pool or not. Whenever a demand is rejected (and so the spare part is not satisfied from the joint spare parts pool), an emergency procedure is instigated (as discussed before), with downtime costs that depend on whose demand is rejected. We assume that the policy of accepting or rejecting demand is such that the long-run average (downtime) costs per time unit are minimized. It follows, based on a classical result of Ha [5], that this policy can be described in terms of a critical level policy.[1] A critical level policy is characterised as follows. According to the different downtime costs, demand of a certain player is accepted if and only if the (total) inventory level (of the joint spare parts pool) is above a certain critical level only. In this way, some spare parts are reserved for the more critical players, e.g., for players with relatively high downtime costs. So, in fact, there is some kind of stratification that determines when players are allowed to make use of the joint spare parts pool. For that reason, we refer to this optimal form of pooling by *stratified pooling*. We denote the minimal long-run average (downtime) costs per time unit for coalition $S \subseteq N$ by $c^\theta(S)$ and set $c^\theta(\emptyset) = 0$. The associated game (N, c^θ) will be called a *stratified pooling* game.

Example 1. *Let $\theta \in \Theta$ be a spare parts situation with $N = \{1,2,3\}$, $I = (1,1,1), d = (3,2,1)$, $\lambda = (1,1,1)$ and $\mu = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For the grand coalition, the optimal critical level policy is of the form where player 1 can satisfy demand as long as $I_N \geq 1$, player 2 can satisfy demand as long as $I_N \geq 2$, and player 3 can satisfy demand as long as $I_N \geq 3$. According to this optimal policy, one can construct a corresponding Markov chain (see Figure 1).*

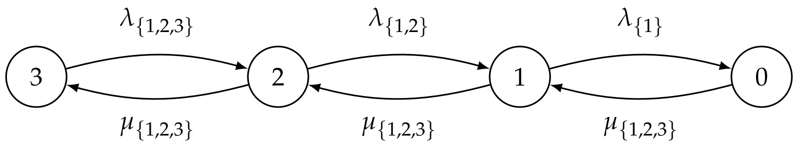

Figure 1: Underlying Markov chain for grand coalition

Based on this Markov chain it is easy to determine the steady state probabilities of state 0 $\left(\frac{16}{67}\right)$, state 1 $\left(\frac{24}{67}\right)$, state 2 $\left(\frac{18}{67}\right)$ and state 3 $\left(\frac{9}{67}\right)$. According to these steady state probabilities, one

[1]See Theorem 1 of Ha [5] with $h(x) = 0$ for $x \leq I_S$ and $h(x) = \infty$ otherwise.

can determine the minimal long-run average costs per time unit as follows

$$c(N) = \frac{16}{67} \cdot (\lambda_1 \cdot d_1 + \lambda_2 \cdot d_2 + \lambda_3 \cdot d_3) + \frac{24}{67} \cdot (\lambda_2 \cdot d_2 + \lambda_3 \cdot d_3) + \frac{18}{67} \cdot (\lambda_3 \cdot d_3) = 2\frac{52}{67}.$$

Similarly, one can determine the costs of the other coalitions (see Table 1 for these values). Note that $x = \left(1\frac{52}{67}, 1, 0\right) \in \mathscr{C}(N, c^\theta)$, *i.e., the core of game* (N, c^θ) *is non-empty.* ◇

Table 2: Corresponding costs per coalition

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c^\theta(M)$	0	2	$1\frac{1}{3}$	$\frac{2}{3}$	$2\frac{4}{5}$	2	$1\frac{3}{5}$	$2\frac{52}{67}$

Remark 1. *Recall that our aim is to show core non-emptiness for stratified pooling games. This result is not longer guaranteed whenever coalitions would (naively) apply a (non-optimal) full-pooling policy. For instance, consider* $\theta \in \Theta$ *with* $N = \{1, 2\}$, $I = (1, 1)$, $d = (1, 4)$, $\lambda = (5, 1)$, *and* $\mu = (1, 1)$. *Under full pooling we would have* $c^\theta(\{1\} + c^\theta(\{2\}) = 4\frac{1}{6} + 2 < 6\frac{3}{13} = c^\theta(\{1, 2\})$, *i.e., the core is empty. However, if an optimal policy is applied, which dictates that player 1 can satisfy demand as long as* $I_N = 2$ *and player 2 can satisfy demand as long as* $I_N = 1$, *we have* $c^\theta(\{1, 2\}) = 5\frac{2}{11}$ *and so the core is non-empty again.*

3.3 MDP formulation

In line with Ha [5] the (accept or reject demand) decision problem per coalition can be considered as a (discrete time) Markov decision process (MDP) as well. This is allowed since the decision problem per coalition can be recognized as a semi-Markov decision problem, which can be converted to an equivalent MDP by applying *uniformization* (see, e.g., Lippman [12]). For that, we add fictitious transitions of a state to itself to ensure that the total rate out of a state is equal for all states, the so-called *uniformization rate*. Then, we consider the embedded discrete-time MDP by looking at the system only at transition instants, which occur according to a Poisson process, with as rate the uniformization rate. This modelling technique turns out to be very useful. Let $\theta \in \Theta$ and $S \subseteq N$ with $S \neq \emptyset$. In what follows, we present this corresponding MDP.

3.3.1 State and Action Spaces

We define the state space to be $\mathscr{Y}^S = \{0, 1, \ldots, I_S\}$ with $i \in \mathscr{Y}^S$ representing the number of spare parts in stock of coalition S and the action space to be $\mathscr{A}^S(y) = \{\mathscr{A}_i^S(y)\}_{i \in S}$ with $\mathscr{A}_i^S(y) = \{0, 1\}$ for all $i \in N$ and all $y > 0$ and $\mathscr{A}_i^S(y) = \{0\}$

otherwise. In state $y \in \mathscr{Y}^S$, action 1 corresponds with the acceptance of a demand at a player, while action 0 corresponds with the rejection of such a demand.

3.3.2 Costs and transition probabilities

Let $\gamma = \sum_{i \in N} [\lambda_i + \mu_i]$. We will use γ as the uniformization rate, which is independent of S. In addition, let $\lambda_i^* = \lambda_i / \gamma$ and $\mu_i^* = \mu_i / \gamma$ for all $i \in N$. Now, $C^S(y, a)$ denotes the expected costs collected over a single (uniformized) time epoch, given that the system begins the period in state $y \in \mathscr{Y}^S$ and action $a = (a_i)_{i \in S} \in \mathscr{A}^S(y)$ is taken. For our situation, we have

$$C^S(y,a) = \sum_{i \in S} \lambda_i^* \cdot (1 - a_i) \cdot d_i \qquad \text{for all } y \in \mathscr{Y}^S \text{ and all } a \in \mathscr{A}^S(y).$$

In addition, let $p^S(y'|y,a)$ denote the one-stage transition probability from state $y \in \mathscr{Y}^S$ to $y' \in \mathscr{Y}^S$ under action $a = (a_i)_{i \in S} \in \mathscr{A}^S(y)$. We have

$$p^S(y'|y,a) = \begin{cases} \sum\limits_{i \in S} \lambda_i^* \cdot a_i & \text{if} \quad y' = y - 1, y > 0 \\ \sum\limits_{i \in S} \mu_i^* & \text{if} \quad y' = y + 1, y < I_S \\ 1 - \sum\limits_{i \in S} \left[\lambda_i^* \cdot a_i + \mu_i^*\right] & \text{if} \quad y' = y < I_S \\ 1 - \sum\limits_{i \in S} \left[\lambda_i^* \cdot a_i\right] & \text{if} \quad y' = y = I_S \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in \mathscr{Y}^S$ and all $a \in \mathscr{A}^S(y)$.

3.3.3 Value function and equivalence

Now, we present the value function in a form suitable for this article. Recall that the proofs of the lemmas are relegated to the appendix.

Lemma 1. *Let $\theta \in \Theta$ and $S \subseteq N$. Then, for all $y \in \mathscr{Y}^S$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_{t+1}^S(y) = \sum_{i \in S} \left[\lambda_i^* \min_{l \in \{0, \min\{y,1\}\}} \left\{V_t^S(y-l) + (1-l)d_i\right\} + \mu_i^* V_t^S(\min\{y+1, I_S\})\right] + \left(1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^S(y)$$

with $V_0^S(y) = 0$ for all $y \in \mathscr{Y}^S$.

Note that the result of Lemma 1 follows by some rewriting of the value function (as represented in (1)) and using that decision $a \in \mathscr{A}^S(y)$ can be decomposed into decisions per player. The formulation of the value function in Lemma 1 can be interpreted in the following way. With probability λ_i^* there is a demand arrival and, except for the state with no spare parts in stock, there is a possibility to (i) accept demand $(l = 1)$ or (ii) reject demand $(l = 0)$ and incur related costs d_i. Based on the first decision $(l = 1)$ there will be a transition to the state with one spare part less in stock and based on the second decision $(l = 0)$ there will be a transition back to the same state. With probability μ_i^* there is, except for the state with no outstanding repair orders, a repair completion, which leads to a transition to the state with one spare part more in stock. With probability $1 - \sum_{i \in S}(\lambda_i^* + \mu_i^*)(\geq 0)$ there is a dummy transition back to the same state, ensuring that the probabilities sum to one.

Example 2. *Consider the situation of Example 1. Observe that $\gamma = 4\frac{1}{2}$, $\lambda_i^* = \frac{2}{9}$ for all $i \in N$, and $\mu_i^* = \frac{1}{9}$ for all $i \in N$. For instance, for coalition $M = \{1,2\}$ with $I_{\{1,2\}} = 2$, the value function for all $t \in \mathbb{N} \cup \{0\}$ is given by*

$$V_{t+1}^{\{1,2\}}(0) = \sum_{i=1}^{2} \left(\frac{2}{9} \left(V_t^{\{1,2\}}(0) + d_i \right) + \frac{1}{9} V_t^{\{1,2\}}(1) \right) + \frac{1}{3} V_t^{\{1,2\}}(0)$$

$$V_{t+1}^{\{1,2\}}(1) = \sum_{i=1}^{2} \left(\frac{2}{9} \min_{l \in \{0,1\}} \left\{ V_t^{\{1,2\}}(1-l) + (1-l)d_i \right\} + \frac{1}{9} V_t^{\{1,2\}}(2) \right) + \frac{1}{3} V_t^{\{1,2\}}(1)$$

$$V_{t+1}^{\{1,2\}}(2) = \sum_{i=1}^{2} \left(\frac{2}{9} \min_{l \in \{0,1\}} \left\{ V_t^{\{1,2\}}(2-l) + (1-l)d_i \right\} + \frac{1}{9} V_t^{\{1,2\}}(2) \right) + \frac{1}{3} V_t^{\{1,2\}}(2).$$

Note that for zero spare parts in stock, it is not possible to accept demand and for two spare parts in stock, a possible replenishment has no effect on the inventory level. $\diamond$

Finally, we define g^S as the minimal long-run average costs per time *epoch* of the MDP. Now, we show that there is a direct relation between g^S and the original minimal long-run average costs per *time unit* of coalition S.

Lemma 2. *Let $\theta \in \Theta$ and $S \subseteq N$ with $S \neq \emptyset$. Then*

$$c^\theta(S) = \gamma \cdot g^S = \gamma \cdot \lim_{t \to \infty} \frac{V_t^S(y)}{t} \quad \text{for all } y \in \mathscr{Y}^S.$$

Example 3. *Consider the situation of Example 2. For coalition $M = \{1,2\}$ it holds that $\lim_{t\to\infty} V_t^{\{1,2\}}(0)/t = \frac{28}{45}$. This value can be interpreted as the minimal long-run average costs per time epoch of the corresponding MDP. Multiplying this value with the uniformization rate $\gamma(= 4\frac{1}{2})$ yields $\frac{126}{45}(= 2\frac{4}{5} = c^\theta(\{1,2\}))$, which can be recognised as the minimal long-run average costs per time unit of coalition M.* $\diamond$

4 Core non-emptiness of stratified pooling games

In the remainder of this article, we focus on core non-emptiness for stratified pooling games. In the literature, there exists a well-known sufficient and necessary condition for core non-emptiness due to Bondareva [3] and Shapley [19]. They formulate this (sufficient and necessary) condition in terms of balanced maps. In order to describe this condition, we need to introduce some definitions. Let $N \subseteq \mathbb{N}$ be a finite player set. We call a map $\kappa : 2^N \backslash \{\emptyset\} \to [0, 1]$ a *balanced map* for N if

$$\sum_{S \in 2^N:\, i \in S} \kappa_S = 1 \quad \text{for all } i \in N.$$

A collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ is called *balanced* if there exists a balanced map κ for which $\kappa_S > 0$ for all $S \in \mathscr{B}$ and $\kappa_S = 0$ otherwise. Moreover, a collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ is called *minimal balanced* if there exists no proper subcollection of $\mathscr{B}$ that is balanced as well. An advantage of minimal balanced collections is that for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ there exists exactly one associated balanced map κ (Peleg and Sudhölter [15]). For this balanced map it holds that $\kappa_S \in \mathbb{Q}$ for all $S \in \mathscr{B}$ (Norde and Reijnierse [14]). A game (N, c) is called *balanced* if for every minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ with associated balanced map κ it holds that

$$\sum_{S \in \mathscr{B}} \kappa_S \cdot c(S) \geq c(N).$$

Now, we are able to present a sufficient and necessary condition for core non-emptiness due to Bondareva [3] and Shapley [19].

Theorem 2. *A game (N, c) is balanced if and only if $\mathscr{C}(N, c) \neq \emptyset$.*

Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. We define $\alpha \in \mathbb{N}$ as the *smallest* integer for which $\kappa_S \cdot \alpha \in \mathbb{N}$ for all $S \in \mathscr{B}$ and use $b_S = \kappa_S \cdot \alpha$ for all $S \in \mathscr{B}$ as a shorthand notation. Note that for these new definitions, we suppress the dependency on $\mathscr{B}$ of α, b_S, and κ_S. So, in order to show balancedness for our stratified pooling game (N, c^θ), it suffices to check if for each $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ it holds that

$$\sum_{S \in \mathscr{B}} b_S \cdot c^\theta(S) \geq \alpha \cdot c^\theta(N). \tag{3}$$

In the remainder of this section, we prove balancedness for our game by showing that (3) holds for each minimal balanced collection. This proof consists of several steps, and to facilitate understanding of the steps we first informally summarize them:

0. **Definition** of an MDP for each coalition S such that $c^\theta(S) = \gamma \cdot \lim_{t\to\infty} V_t^S(y)/t$, where V^S is the value function corresponding to the MDP (see Section 3.3).

1. **Copy** of each coalition S to obtain labeled coalitions (S,k) for $k \in \{1,\ldots,b_S\}$ and associated value function $V^{S,k}$. Express the left-hand size of (3) in terms of $V^{S,k}$.

2. **Combination** of the value functions $V^{S,k}$ for all labeled coalitions (S,k) into a single value function $V^{\mathscr{B}}$. The combination of this new value function is semi-cartesian: each individual value function $V^{S,k}$ is retained in $V^{\mathscr{B}}$ along with all its dynamics, while the transitions (due to demand arrivals or repair completions) are coupled across the individual value functions.

3. **Relaxation** of the possible transition actions in $V^{\mathscr{B}}$ to obtain $\widehat{V}^{\mathscr{B}}$. This latter value function corresponds to a situation where demand of a labeled coalition can be satisfied using inventory of *any* labeled coalition and where a repair completion of a labeled coalition can be used to increase the inventory of *any* labeled coalition.

4. **Anonimization** of the state space belonging to $\widehat{V}^{\mathscr{B}}$ to obtain an MDP that only keeps track of the total inventory of all labeled coalitions together, with associated value function V^α. In this MDP demands arrive in batches of size α, and each repair completion simultaneously returns (at most) α parts to inventory.

5. **Uncopy** of value function V^α into α-times the value function V^N, which is the value function of the grand coalition.

We next discuss steps 1-5 in detail and present a **conclusion** which proves (3).

1. Copy. For each minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$, we introduce another set $\mathscr{L}$ that contains for each $S \in \mathscr{B}$ exactly b_s labeled *copies* of coalition S.

Definition 1. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, we define*

$$\mathscr{L} = \Big\{ \ (S,k) \ \mid \ S \in \mathscr{B}, k \in \{1,2,\ldots,b_S\} \Big\}.$$

Example 4. *Let $\theta \in \Theta$ with $|N| = 4$ and $\mathscr{B} = \{\{1\},\{2,3\},\{2,4\},\{3,4\}\}$ be a minimal balanced collection with unique weights $\kappa_{\{1\}} = 1, \kappa_{\{2,3\}} = \kappa_{\{2,4\}} = \kappa_{\{3,4\}} = \frac{1}{2}$. Hence, $\alpha = 2$, and so $\mathscr{L} = \{(\{1\},1),(\{1\},2),(\{2,3\},1),(\{2,4\},1),(\{3,4\},1)\}$.*

The labeled copies will be called *labeled coalitions*. For each labeled coalition $(S,k) \in \mathscr{L}$ we denote the value function by $V^{S,k}$ and the initial inventory level by $I_{S,k}$. In addition, we rewrite the labeled coalitional values (corresponding to a minimal balanced collection) as stated in Lemma 2, i.e., as limits of value functions.

Lemma 3. *For every $\theta \in \Theta$ it holds for any minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ that*

$$\sum_{S \in \mathscr{B}} b_S \cdot c^{\theta}(S) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \cdot \sum_{S \in \mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(I_{S,k}).$$

2. Combination. We show that for any minimal balanced collection $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ we can construct a *combined* value function (of some unspecified MDP) with a state space that keeps track of the inventory level of every labeled coalition $(S,k) \in \mathscr{L}$, an action space that consists of all possible actions per labeled coalition $(S,k) \in \mathscr{L}$ given its inventory level, and for which the related costs coincide with $\sum_{S \in \mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(I_{S,k})$ for all $t \in \mathbb{N} \cup \{0\}$. In order to do so, we first introduce a new state space.

Definition 2. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, we define*

$$\mathscr{Y}^{\mathscr{B}} = \left\{ \; (r_z)_{z \in \mathscr{L}} \;\middle|\; r_z \in \{0, 1, \ldots, I_z\} \;\; \forall z \in \mathscr{L} \right\}.$$

Secondly, we will introduce a new action space.

Definition 3. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ we define*

$$\mathscr{A}_{i,-}^{\mathscr{B}}(r) = \left\{ \; (l_z)_{z \in \mathscr{L}} \;\middle|\; \begin{array}{cc} l_z \in \{0, \min\{1, r_z\}\} & \forall z \in \mathscr{L} : i \in S \\ l_z = 0 & \forall z \in \mathscr{L} : i \notin S \end{array} \right\}$$

$$\mathscr{A}_{i,+}^{\mathscr{B}}(r) = \left\{ \; (l_z)_{z \in \mathscr{L}} \;\middle|\; \begin{array}{cc} l_z = \min\{1, I_z - r_z\} & \forall z \in \mathscr{L} : i \in S \\ l_z = 0 & \forall z \in \mathscr{L} : i \notin S \end{array} \right\}.$$

Subsequently, we introduce the new value function. We use that $||\cdot||_1$ is the L^1 norm.

Definition 4. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$, we define the value function as*

$$V_{t+1}^{\mathscr{B}}(r) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \mathscr{A}_{i,-}^{\mathscr{B}}(r)} \left\{ (\alpha - ||l||_1) d_i + V_t^{\mathscr{B}}(r - l) \right\} + \mu_i^* \min_{l \in \mathscr{A}_{i,+}^{\mathscr{B}}(r)} \left\{ V_t^{\mathscr{B}}(r + l) \right\} \right].$$

with $V_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{Y}^{\mathscr{B}}$.

The new value function $V^{\mathscr{B}}$ can be interpreted in the following way. With probability λ_i^* there is a demand for all labeled coalitions $(S,k) \in \mathscr{L}$ for which $i \in S$. Each such labeled coalition (S,k) has, except for the case with $r_{S,k} = 0$, the possibility to accept the single demand ($l_{S,k} = 1$), and always the possibility to reject the single demand ($l_{S,k} = 0$). For all (other) labeled coalitions $(S,k) \in \mathscr{L}$ for which $i \notin S$ it holds that there is no demand arrival and so $l_{S,k} = 0$. Based on these decisions, total costs equal

$(\alpha - ||l||_1)d_i$ and one transits to $r - l$. With probability μ_i^* there is a repair completion for each labeled coalition $(S,k) \in \mathscr{L}$ for which $i \in S$. For each labeled coalition $(S,k) \in \mathscr{L}$ with $i \in S$ and $r_{S,k} < I_{S,k}$ one accepts the spare part ($l_z = 1$). However, for each labeled coalition $(S,k) \in \mathscr{L}$ with $i \in S$ and $r_{S,k} = I_{S,k}$ the spare part is rejected ($l_{S,k} = 0$) as inventory level $I_{S,k}$ has been reached[2]. For all (other) labeled coalitions $(S,k) \in \mathscr{L}$ for which $i \notin S$ it holds that there is no repair completion and so $l_{S,k} = 0$. Based on the decisions made, one transits to state $r + l$.

Example 5. *Consider the situation of Example 3 and $\mathscr{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$. Observe that $\mathscr{B}$ is a minimal balanced collection with $\kappa(\{1,2\}) = \kappa(\{1,3\}) = \kappa(\{2,3\}) = \frac{1}{2}$. So, $\alpha = 2$ and $\mathscr{L} = \{(\{1,2\},1),(\{1,3\},1),(\{2,3\},1)\}$. Then, we have a state space*

$$\mathscr{Y}^{\mathscr{B}} = \{0,1,2\}^{\mathscr{L}}.$$

Moreover, for $r = (r_{(\{1,2\},1)}, r_{(\{1,3\},1)}, r_{(\{2,3\},1)}) = (1,2,0) \in \mathscr{Y}^{\mathscr{B}}$, we have an action space

$$\mathscr{A}^{\mathscr{B}}_{1,-}((1,2,0)) = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}.$$

Note that the elements in this action space represent the possible actions that can be taken by all possible labeled coalitions $(\{1,2\},1)$, $(\{1,3\},1)$, and $(\{2,3\},1)$ whenever there is a demand for labeled coalition $(\{1,2\},1)$ and $(\{1,3\},1)$. As both labeled coalitions have at least one spare part in stock, they can both accept this demand, one of them can except, or both can reject. Similarly, for action space $(0,1,1) \in \mathscr{Y}^{\mathscr{B}}$, we have

$$\mathscr{A}^{\mathscr{B}}_{1,-}((0,1,1)) = \{(0,0,0),(0,1,0)\}.$$

The action space illustrates that the spare part of labeled coalition $(\{2,3\},1)$ cannot be used by the labeled coalitions $(\{1,2\},1)$ and $(\{1,3\},1)$. $\diamond$

Now, we are able to show for all time moments the equivalence between the costs of the new value function and $\sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(I_{S,k})$.

Lemma 4. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N}_+ \cup \{0\}$ it holds that*

$$\sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}) = V_t^{\mathscr{B}}(r).$$

[2] A possible interpretation could be to see this repair completion in fact as a *possible* repair completion where it is a repair completion if $r_{S,k} < I_{S,k}$ and not if $r_{S,k} = I_{S,k}$.

3. Relaxation. Note that the action space of $V^{\mathscr{B}}$ is restricted. For instance, upon a demand arrival of player $i \in S$, it is not possible to accept a single demand for labeled coalition $(S,k) \in \mathscr{L}$ for which $i \in S$ and $r_{S,k} = 0$, while other labeled coalitions $(S,k) \in \mathscr{L}$ for which $i \notin S$ may still be able to accept them. A similar reasoning holds for repair completion. It is not possible to accept a spare part for any labeled coalition $(S,k) \in \mathscr{L}$ with $i \in S$ for which $r_{S,k} = I_{S,k}$, while other not fully replenished labeled coalitions may be able to accept. We introduce a value function that incorporates these extended possibilities. So, we introduce a value function (related to some unspecified MDP), that coincides with the value function of Definition 4, except for a *relaxed* action space. In order to do so, we first need to introduce this relaxed action space.

Definition 5. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ we define*

$$\widehat{\mathscr{A}_{i,-}^{\mathscr{B}}}(r) = \left\{ (l_z)_{z\in\mathscr{L}} \;\middle|\; l_z \in \mathbb{N} \cup \{0\} \;\; \forall z \in \mathscr{L},\; \sum_{z\in\mathscr{L}} l_z \leq \alpha,\; r - l \in \mathscr{Y}^{\mathscr{B}} \right\}$$

$$\widehat{\mathscr{A}_{i,+}^{\mathscr{B}}}(r) = \left\{ (l_z)_{z\in\mathscr{L}} \;\middle|\; l_z \in \mathbb{N} \cup \{0\} \;\; \forall z \in \mathscr{L},\; \sum_{z\in\mathscr{L}} l_z \leq \alpha,\; r + l \in \mathscr{Y}^{\mathscr{B}} \right\}$$

The following result is a direct consequence of relaxing the action space. The proof is straightforward and for this reason omitted (rather than relegated to the appendix).

Lemma 5. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $i \in N$ it holds that $\mathscr{A}_{i,-}^{\mathscr{B}}(r) \subseteq \widehat{\mathscr{A}_{i,-}^{\mathscr{B}}}(r)$ and $\mathscr{A}_{i,+}^{\mathscr{B}}(r) \subseteq \widehat{\mathscr{A}_{i,+}^{\mathscr{B}}}(r)$.*

Example 6. *Consider the situation and minimal balanced collection of Example 5. Then*

$$\widehat{\mathscr{A}_{1,-}^{\mathscr{B}}}((1,2,0)) = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,2,0)\}.$$

Note that in comparison to action space $\widehat{\mathscr{A}_{1,-}^{\mathscr{B}}}((1,2,0))$, we now have one additional action, namely $(0,2,0)$, which represents that demand for labeled coalitions $(\{1,2\},1)$ and $(\{1,3\},1)$ is taken from the stock of labeled coalition $(\{1,3\},1)$ only. Similarly, we have

$$\widehat{\mathscr{A}_{1,-}^{\mathscr{B}}}((0,1,1)) = \{(0,0,0),(0,1,0),(0,0,1),(0,1,1)\}.$$

Note that in comparison to action space $\widehat{\mathscr{A}_{1,-}^{\mathscr{B}}}((0,1,1))$, we now have two additional actions. $\diamond$

Now, we present the value function with this relaxed action space.

Definition 6. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$, we define*

$$\widehat{V}_{t+1}^{\mathscr{B}}(r) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \widehat{\mathscr{A}_{i,-}^{\mathscr{B}}}(r)} \left\{ (\alpha - ||l||_1) d_i + \widehat{V}_t^{\mathscr{B}}(r - l) \right\} + \mu_i^* \min_{l \in \widehat{\mathscr{A}_{i,+}^{\mathscr{B}}}(r)} \left\{ \widehat{V}_t^{\mathscr{B}}(r + l) \right\} \right].$$

with $\widehat{V}_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{Y}^{\mathscr{B}}$.

Incorporating a relaxed action space in the new value function leads to related costs that are smaller than or equal to the original costs of the value function. The proof is straightforward (by induction on t) and therefore omitted (in the appendix).

Lemma 6. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_t^{\mathscr{B}}(r) \geq \widehat{V}_t^{\mathscr{B}}(r).$$

4. Anonimization. Note that the costs of $\widehat{V}_t^{\mathscr{B}}$ are the same for all states $r, r' \in \mathscr{Y}^{\mathscr{B}}$ for which $||r||_1 = ||r'||_1$. As a consequence, the decisions made in these states exhibit a similar equivalence. This implies that for every minimal balanced collection, one can construct another value function with equal costs, in which the state space depends on the total inventory of all labeled coalitions together only and the action space depends on the total number of accepted (or rejected) demand for spare parts upon demand arrival or repair completion only. Now, we introduce this value function (which is related to some unspecified MDP for which the belongings and decisions of the labeled coalitions are *anonimized*) and show cost-equivalence with the one of Definition 6.

Definition 7. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. Then, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ we define*

$$V_{t+1}^{\alpha}(j) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \{0,..,\min\{\alpha, j\}\}} \left\{ (\alpha - l) d_i + V_t^{\alpha}(j - l) \right\} + \mu_i^* \min_{l \in \{0,..,\min\{\alpha, \alpha \cdot I_N - j\}\}} V_t^{\alpha}(j + l) \right]$$

with $V_0^{\alpha}(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$.

Lemma 7. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathscr{Y}^{\mathscr{B}}$ it holds that*

$$\widehat{V}_t^{\mathscr{B}}(r) = V_t^{\alpha}(||r||_1) \qquad \textit{for all } t \in \mathbb{N} \cup \{0\}.$$

5. Uncopy. Now, we identify some interesting properties of V^{α}.

Lemma 8. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $t \in \mathbb{N} \cup \{0\}$ it holds that*

(i) $V_t^\alpha(j) \geq V_t^\alpha(j+1)$ *for all* $j \in \{0, 1, \ldots, \alpha \cdot I_N - 1\}$;

(ii) $V_t^\alpha(j) + V_t^\alpha(j+2) \geq 2 \cdot V_t^\alpha(j+1)$ *for all* $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$;

(iii) $V_t^\alpha(k+j) + V_t^\alpha(k+j+2) = 2 \cdot V_t^\alpha(k+j+1)$ *for all* $j \in \{0, 1, \ldots, \alpha - 2\}$ *and all* $k \in \{0, \alpha, 2\alpha, \ldots, (I_N - 1)\alpha\}$.

The first property states that the costs decrease in the total number of spare parts in stock. As a direct consequence, it is optimal for the repair completion part of the value function to choose an action that increases the state most. The second property states that the marginal change in costs is increasing in the total number of spare parts in stock, i.e., V^α is convex. By exploiting the first two properties, one can show that the third property holds true. The third property implies that it is optimal for all states that are multiples of α to choose the action that accepts all demand, i.e., α spare parts, or nothing upon demand arrival. From this, we can conclude that the states of V^α that are multiples of α depend on the states of V^α that are multiples of α only. This allows us to *uncopy* value function V^α into $\alpha - \textit{times}$ value function V^N.

Lemma 9. *Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. For all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that*

$$V_t^\alpha(j) = \alpha \cdot V_t^N \left(\frac{j}{\alpha} \right)$$

Conclusion. Now, we integrate the previous steps to demonstrate validity of (3).

Theorem 3. *Stratified pooling games are balanced.*

Proof : Let $\theta \in \Theta$ and (N, c^θ) be the associated stratified pooling game and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. In addition, let $\widehat{r} = (I_{S,k})_{(S,k) \in \mathscr{L}}$. Then, observe that

$$\begin{aligned}
\sum_{S \in \mathscr{B}} b_S \cdot c^\theta(S) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \sum_{S \in \mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(I_{S,k}) \geq \gamma \cdot \lim_{t \to \infty} \frac{\widehat{V}_t^{\mathscr{B}}(\widehat{r})}{t} &= \gamma \cdot \lim_{t \to \infty} \frac{V_t^\alpha(\alpha \cdot I_N)}{t} \\
&= \gamma \cdot \lim_{t \to \infty} \alpha \cdot \frac{V_t^N(I_N)}{t} \\
&= \alpha \cdot c^\theta(N).
\end{aligned}$$

The first equality holds by Lemma 3. The inequality holds by Lemma 4 and Lemma 6. The second equality holds by Lemma 7 and the fact that $\sum_{(S,k) \in \mathscr{L}} I_{S,k} = \alpha \cdot I_N$. The third equality holds by Lemma 9. The last equality holds by taking $\alpha \in \mathbb{N}_+$ outside the limit (which is allowed as it is a constant) and subsequently applying Lemma 2. Finally, inequality (3) is satisfied and so stratified pooling games are balanced. $\square$

Based on Theorem 2, which states that a game has a non-empty core if and only if it is balanced, the next result follows immediately.

Corollary 1. *Stratified pooling games have a non-empty core.*

5 Conclusions

We studied an environment with several service providers that are located geographically close together. These service providers keep spare parts in stock to protect for downtime of their high-tech machines and face different downtime costs per stock-out. Service providers can cooperate by forming a joint spare parts pool and we studied the allocation of the joint costs to the individual service providers by studying an associated cooperative game. In the extant literature, the joint spare parts inventory is typically controlled by a suboptimal full-pooling policy. This may lead to an empty core of the associated cooperative game. We showed possible emptiness of the core under a full-pooling policy in our setting as well. The focus of the paper was then on situations in which we allow service providers to apply an optimal policy. We formulated the associated game, which we call a stratified pooling game, by defining each coalitional value in terms of the minimal long-run average costs of an MDP. We showed, via a 5-step proof, that the core of stratified pooling games is always non-empty. These modelling and proof techniques may be more generically applicable for operations research games where coalitional values correspond to the minimal long-run average costs of MDPs as well.

For further research, we believe the following extensions are of interest. First, one can extend the model, in line with Karsten et al. [6] and Schlicher et al. [17], by assuming that the joint supplier has multiple parallel servers (instead of one single server in which production rates are combined). Another possible extension is the one in which coalitions optimize the number of spare parts to stock, rather than assuming that the inventory per coalition is given. Under this extension, players face a trade off between holding costs and downtime costs under an optimal pooling strategy. Finally, one can extend the model by assuming that players are not located geographically close together, but still want to collaborate (e.g., via lateral transshipments of spare parts).

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Appendix

Proof of Lemma 1

We distinguish between the case $I_S = 0$ and $I_S > 0$.

Case 1. $\quad I_S = 0$.

Let $j = 0 \in \mathscr{Y}^S$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^S(j) &= \min_{a \in \mathscr{A}^S(j)} \left\{ C^S(j,a) + \sum_{y' \in \mathscr{Y}^S} p(y'|j,a) \cdot V_t^S(y') \right\} \\
&= C^S(j, 0^N) + \sum_{y' \in \mathscr{Y}^S} p(y'|j, 0^N) \cdot V_t^S(y') \\
&= \sum_{i \in S} \lambda_i^* \cdot d_i + 1 \cdot V_t^S(j) \\
&= \sum_{i \in S} \lambda_i^* \cdot d_i + \left(\sum_{i \in S} [\lambda_i^* + \mu_i^*] + 1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^S(j) \\
&= \sum_{i \in S} \left[\lambda_i^* \cdot (d_i + V_t^S(j)) + \mu_i^* V_t^S(j) \right] + \left(1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^S(j) \\
&= \sum_{i \in S} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^S(j-l) + (1-l) d_i \right\} + \mu_i^* V_t^S(\min\{j+1, I_S\}) \right] \\
&\qquad + \left(1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^S(j).
\end{aligned}$$

The first equality holds by definition. The second equality holds as action $a = 0^N$ is the only allowed action. The third equality holds by the definition of $C^S(j,a)$ and the fact that one can transit to the current state ($j = 0$) only. In the fourth equality, we add zero. The fifth equality holds by some rewriting. The last equality holds as $\min\{j, 1\} = j = 0$ and $\min\{j+1, I_S\} = I_S = j = 0$ as well.

Case 2. $\quad I_S > 0$.

We distinguish between the subcases $j = 0$, $0 < j < I_S$, and $j = I_S$.

Case 2.a $\quad j = 0$.

Let $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^S(j) &= \min_{a \in \mathscr{A}^S(j)} \left\{ C^S(j,a) + \sum_{y' \in \mathscr{Y}^S} p(y'|j,a) \cdot V_t^S(y') \right\} \\
&= C^S(j, 0^N) + \sum_{y' \in \mathscr{Y}^S} p(y'|j, 0^N) \cdot V_t^S(y')
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in S} \lambda_i^* \cdot d_i + \left(1 - \sum_{i\in S} \mu_i^*\right) \cdot V_t^S(j) + \sum_{i\in S} \mu_i^* \cdot V_t^S(j+1) \\
&= \sum_{i\in S} \lambda_i^* \cdot d_i + \left(\sum_{i\in S} \lambda_i^* + \sum_{i\in N\setminus S} \lambda_i^* + \sum_{i\in N\setminus S} \mu_i^*\right) \cdot V_t^S(j) + \sum_{i\in S} \mu_i^* \cdot V_t^S(j+1) \\
&= \sum_{i\in S} \lambda_i^* \cdot (d_i + V_t^S(j)) + \sum_{i\in S} \mu_i^* \cdot V_t^S(j+1) + \sum_{i\in N\setminus S} [\lambda_i^* + \mu_i^*] \cdot V_t^S(j) \\
&= \sum_{i\in S} \left[\lambda_i^* \min_{l\in\{0,\min\{j,1\}\}} \left\{V_t^S(j-l) + (1-l)d_i\right\} + \mu_i^* V_t^S(\min\{j+1, I_S\})\right] \\
&\qquad + \left(1 - \sum_{i\in S} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^S(j).
\end{aligned}
$$

The first equality holds by definition. The second equality holds as action $a = 0^N$ is the only allowed action. In the third equality, we use the definition of $C^S(j,a)$ and the fact that with probability $1 - \sum_{i\in S} \mu_i^*$ we remain in the same state and with probability $\sum_{i\in S} \mu_i^*$ we transit to state $j+1$. In the fourth equality, we use that $1 = \sum_{i\in N} \left[\lambda_i^* + \mu_i^*\right] = \sum_{i\in S} \lambda_i^* + \sum_{i\in N\setminus S} \lambda_i^* + \sum_{i\in N} \mu_i^*$. The fifth equality holds by some rewriting. The last equality holds as $\min\{j,1\} = j = 0$ and $\min\{j+1, I_S\} = j+1 = 1$.

Case 2.b $0 < j < I_S$.

Let $j \in \mathscr{Y}^S \setminus \{0, I_S\}$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$
\begin{aligned}
V_{t+1}^S(j) &= \min_{a\in\mathscr{A}^S(j)} \left\{C^S(j,a) + \sum_{y'\in\mathscr{Y}^S} p(y'|j,a) \cdot V_t^S(y')\right\} \\
&= \min_{a\in\mathscr{A}^S(j)} \left\{\sum_{i\in S} \lambda_i^*(1-a_i)d_i + \sum_{i\in M} \lambda_i^* a_i V_t^S(j-1) + \left(1 - \sum_{i\in S} [\lambda_i^* a_i + \mu_i^*]\right) V_t^S(j) \right. \\
&\qquad \left. + \sum_{i\in M} \mu_i^* V_t^S(j+1)\right\} \\
&= \min_{a\in\mathscr{A}^S(j)} \left\{\sum_{i\in S} \left[\lambda_i^* \left((1-a_i)(d_i + V_t^S(j)) + a_i V_t^S(j-1)\right) + \mu_i^* V_t^S(j+1)\right] \right. \\
&\qquad \left. + \left(1 - \sum_{i\in S} [\lambda_i^* + \mu_i^*]\right) V_t^S(j)\right\} \\
&= \sum_{i\in S} \left[\lambda_i^* \min_{l\in\{0,\min\{j,1\}\}} \left\{V_t^S(j-l) + (1-l)d_i\right\} + \mu_i^* V_t^S(\min\{j+1, I_S\})\right] \\
&\qquad + \left(1 - \sum_{i\in S} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^S(j).
\end{aligned}
$$

The first equality holds by definition. In the second equality, we use the definition of $C^S(j,a)$, the fact that with probability $\sum_{i\in S} \lambda_i^* a_i$ we transit to state $j-1$, with probability

$1 - \sum_{i \in S} \left[\lambda_i^* a_i + \mu_i^*\right]$ we remain in the same state, and with probability $\sum_{i \in S} \mu_i^*$ we transit to state $j+1$. In the third equality, we did some rewriting and used $a_i = a_i - 1 + 1$. The last equality holds as $\min\{j, 1\} = 1$, $\min\{j+1, I_S\} = j+1$, and the fact that a minimum of a sum of independent terms equals the sum of all these individual terms evaluated at their minimum.

Case 2.c $\quad j = I_S$.

Let $j = I_S \in \mathscr{Y}^S$ and $t \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
V_{t+1}^S(j) &= \min_{a \in \mathscr{A}^S(j)} \left\{ C^S(j,a) + \sum_{y' \in \mathscr{Y}^S} p(y'|j,a) \cdot V_t^S(y') \right\} \\
&= \min_{a \in \mathscr{A}^S(j)} \left\{ \sum_{i \in S} \lambda_i^* (1-a_i) d_i + \sum_{i \in S} \lambda_i^* a_i V_t^M(j-1) + \left(1 - \sum_{i \in S} \lambda_i^* a_i\right) V_t^S(j) \right\} \\
&= \min_{a \in \mathscr{A}^S(j)} \left\{ \sum_{i \in S} \lambda_i^* (1-a_i) d_i + \sum_{i \in S} \lambda_i^* a_i V_t^S(j-1) + \left(\sum_{i \in S} [\lambda_i^*(1-a_i) + \mu_i^*] \right.\right. \\
&\qquad \left.\left. + \sum_{i \in N \setminus S} [\lambda_i^* + \mu_i^*] \right) V_t^S(j) \right\} \\
&= \min_{a \in \mathscr{A}^S(j)} \left\{ \sum_{i \in S} \lambda_i^* \left[(1-a_i)(d_i + V_t^S(j)) + a_i V_t^S(j-1) + \mu_i^* V_t^S(j) \right] + \right. \\
&\qquad \left. \sum_{i \in N \setminus S} [\lambda_i^* + \mu_i^*] V_t^S(j) \right\} \\
&= \sum_{i \in S} \left[\lambda_i^* \min_{l \in \{0, \min\{j,1\}\}} \left\{ V_t^S(j-l) + (1-l) d_i \right\} + \mu_i^* V_t^S(\min\{j+1, I_S\}) \right] \\
&\qquad + \left(1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*]\right) \cdot V_t^S(j).
\end{aligned}$$

The first equality holds by definition. In the second equality, we use the definition of $C^S(j,a)$, the fact that with probability $\sum_{i \in S} \lambda_i^* a_i$ we transit to state $j-1$, and with probability $1 - \sum_{i \in S} \lambda_i^* a_i$ we remain in the same state. In the third equality, we use that $1 = \sum_{i \in N} \left[\lambda_i^* + \mu_i^*\right]$. The fourth equality holds by some rewriting. The last equality holds as $\min\{j, 1\} = 1$, $\min\{j+1, I_S\} = I_S$ and the fact that a minimum of a sum of independent terms equals the sum of all these individual terms evaluated at their minimum. This concludes the proof. $\square$

Proof of Lemma 2

The first equality holds by uniformization, which is allowed if transition rates are bounded and the MDP is unichain (see Puterman [16, p.568]). Notice that interarrival times of demands as well as repair times are exponentially distributed with rates that are bounded from above. In addition, for every stationary policy, the state where the inventory level equals the base stock level is accessible from every state and so, the transition probability matrix consists of a single recurrent class. Hence, the MDP is unichain. With respect to the second equality, observe that state space $\mathscr{Y}^S$ and action space $\mathscr{A}^S$ of the MDP are finite. In addition, under stationary policy $f = (f_i(y))_{y\in\mathscr{Y}^S, i\in S}$ with $f_i(y) = 1$ for all $i \in S$ and all $y > 0$ and $f_i(0) = 0$ for all $i \in S$, every state $y \in \mathscr{Y}^S$ is accessible from any state $y' \in \mathscr{Y}^S$ after (possibly) some arrivals and some (one-by-one) repair completions. Hence, the related Markov chain is irreducible. An irreducible Markov chain with finite state space is positive recurrent (see e.g., Modica and Poggiolini [13]). Finally, observe that the long-run average costs per time epoch under policy f are bounded (naturally) by $\sum_{i\in S} \lambda_i^* \cdot d_i$ and as a result of Theorem 1, the second equality follows. This concludes the proof. □

Proof of Lemma 3

Let $\theta \in \Theta$ and $\mathscr{B} \subseteq 2^N \backslash \{\emptyset\}$ be a minimal balanced collection. It holds that

$$\sum_{S\in\mathscr{B}} b_S \cdot c^\theta(S) = \gamma \cdot \sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} \lim_{t\to\infty} \frac{V_t^{S,k}(I_{S,k})}{t} = \gamma \cdot \lim_{t\to\infty} \frac{1}{t} \cdot \sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(I_{S,k}).$$

The first equality holds by exploiting all labeled coalitions, Lemma 2, and the fact that $I_{S,k} \in \mathscr{Y}^{S,k}$ for all $(S,k) \in \mathscr{L}$. The last equality holds as all limits are well-defined and all sums are finite. This concludes the proof. □

Proof of Lemma 4

Proof : This proof is by induction. By definition of the value functions $V_0^{S,k}(y) = 0$ for all $y \in \mathscr{Y}^{S,k}$, and all $S \in \mathscr{B}$ and all $k \in \{1, 2, \ldots, b_S\}$. Similarly, $V_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{R}$ as well. Hence, $\sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_0^{S,k}(r_{S,k}) = V_0^{\mathscr{B}}(r)$ for all $r \in \mathscr{R}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $\sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}) = V_t^{\mathscr{B}}(r)$ for all $r \in \mathscr{R}$. Let $r \in \mathscr{R}$. Now, observe that

$$\begin{aligned}
&\sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} V_{t+1}^{S,k}(r_{S,k}) \\
&= \sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} \left(\sum_{i\in S} \left[\lambda_i^* \min_{l\in\{0,\min\{1,r_{S,k}\}\}} \left\{ V_t^{S,k}(r_{S,k} - l) + (1-l)d_i \right\} \right.\right. \\
&\qquad\qquad \left.\left. + \mu_i^* V_t^{S,k}(\min\{r_{S,k}+1, I_{S,k}\}) \right] + \left(1 - \sum_{i\in S} [\lambda_i^* + \mu_i^*] \right) V_t^{S,k}(y) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S\in\mathscr{B}} \sum_{k=1}^{b_S} \left(\sum_{i\in S} \left[\lambda_i^* \min_{l\in\{0,\min\{1,r_{S,k}\}\}} \left\{ V_t^{S,k}(r_{S,k}-l) + (1-l)d_i \right\} \right.\right.\\
&\qquad \left.\left. +\mu_i^* V_t^{S,k}(\min\{r_{S,k}+1, I_{S,k}\}) \right] + \sum_{i\in N\setminus S} \left[\lambda_i^* V_t^{S,k}(r_{S,k}) + \mu_i^* V_t^{S,k}(r_{S,k}) \right] \right)\\
&= \sum_{i\in N} \left[\lambda_i^* \left(\sum_{S\in\mathscr{B}:i\in S} \sum_{k=1}^{b_S} \min_{l\in\{0,\min\{r_{S,k},1\}\}} \left\{ V_t^{S,k}(r_{S,k}-l) + (1-l)d_i \right\} + \sum_{S\in\mathscr{B}:i\notin S} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}) \right) \right.\\
&\qquad \left. + \mu_i^* \left(\sum_{S\in\mathscr{B}:i\in S} \sum_{k=1}^{b_S} V_t^{S,k}(\min\{r_{S,k}+1, I_{S,k}\}) + \sum_{S\in\mathscr{B}:i\notin S} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}) \right) \right]\\
&= \sum_{i\in N} \left[\lambda_i^* \cdot \left(\min_{l\in\mathscr{A}_{i,-}^{\mathscr{B}}(r)} \left\{ \sum_{S\in\mathscr{B}:i\in S} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}-l_{S,k}) + \sum_{S\in\mathscr{B}:i\notin S} \sum_{k=1}^{b_S} V_t^{S,k}(r_{S,k}) \right.\right.\right.\\
&\qquad \left.\left. \left. +(\alpha - ||l||_1)d_i \right\} \right) + \mu_i^* \cdot \left(\min_{l\in\mathscr{A}_{i,+}^{\mathscr{B}}(r)} \left\{ V_t^{\mathscr{B}}(r+l) \right\} \right) \right]\\
&= \sum_{i\in N} \left[\lambda_i^* \cdot \left(\min_{l\in\mathscr{A}_{i,-}^{\mathscr{B}}(r)} \left\{ V_t^{\mathscr{B}}(r-l) + (\alpha - ||l||_1)d_i \right\} \right) + \mu_i^* \cdot \left(\min_{l\in\mathscr{A}_{i,+}^{\mathscr{B}}(r)} \left\{ V_t^{\mathscr{B}}(r+l) \right\} \right) \right]\\
&= V_{t+1}^{\mathscr{B}}(r).
\end{aligned}
$$

The first equality holds by Lemma 1. The second equality holds as $1 - \sum_{i\in S}[\lambda_i^* + \mu_i^*] = \sum_{i\in N}[\lambda_i^* + \mu_i^*] - \sum_{i\in S}[\lambda_i^* + \mu_i^*] = \sum_{i\in N\setminus S}[\lambda_i^* + \mu_i^*]$. The third equality holds by conditioning on λ_i^* and μ_i^* for all $i \in N$. The fourth equality holds as the sum of minima can be rewritten as one minimum and $\mathscr{A}_{i,+}^{\mathscr{B}}$ and $\mathscr{A}_{i,-}^{\mathscr{B}}$ are defined such that the decisions made for all minima fit. Note that $l_{S,k} = 0$ if $i \notin S$. The fifth equality holds by the induction hypothesis. The last equality holds by Definition 4. □

Proof of Lemma 7

Proof : This proof is by induction. By definition of the value functions $\widehat{V}_0^{\mathscr{B}}(r) = 0$ for all $r \in \mathscr{R}$ and $V_0^{\alpha}(||r||_1) = 0$ for all $r \in \mathscr{R}$. Hence, $\widehat{V}_0^{\mathscr{B}}(r) = V_0^{\alpha}(||r||_1)$ for all $r \in \mathscr{R}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $\widehat{V}_t^{\mathscr{B}}(r) = V_t^{\alpha}(||r||_1)$ for all $r \in \mathscr{R}$. Let $r \in \mathscr{R}$. Now, observe that

$$
\begin{aligned}
\widehat{V}_{t+1}^{\mathscr{B}}(r) &= \sum_{i\in N} \lambda_i^* \min_{l\in\widehat{\mathscr{A}}_{i,-}^{\mathscr{B}}(r)} \left\{ (\alpha - ||l||_1)d_i + \widehat{V}_t^{\mathscr{B}}(r-l) \right\} + \sum_{i\in N} \mu_i^* \min_{l\in\widehat{\mathscr{A}}_{i,+}^{\mathscr{B}}(r)} \widehat{V}_t^{\mathscr{B}}(r+l)\\
&= \sum_{i\in N} \lambda_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,||r||_1\}\}} \left\{ \min_{l\in\widehat{\mathscr{A}}_{i,-}^{\mathscr{B}}(r):||l||_1=z} \left\{ (\alpha - z)d_i + \widehat{V}_t^{\mathscr{B}}(r-l) \right\} \right\}\\
&\quad + \sum_{i\in N} \mu_i^* \min_{z\in\{0,1,\ldots,\min\{\alpha,\alpha\cdot I_N-||r||_1\}\}} \left\{ \min_{l\in\widehat{\mathscr{A}}_{i,+}^{\mathscr{B}}(r):||l||_1=z} \widehat{V}_t^{\mathscr{B}}(r+l) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i\in N} \lambda_i^* \min_{z\in\{0,1,\dots,\min\{\alpha,||r||_1\}\}} \{(\alpha - z)d_i + V_t^\alpha(||r||_1 - z)\} \\
&\quad + \sum_{i\in N} \mu_i^* \min_{z\in\{0,1,\dots,\min\{\alpha,\alpha\cdot I_N - ||r||_1\}\}} \left\{ V_t^\alpha(||r||_1 + z) \right\} \\
&= V_{t+1}^\alpha(||r||_1).
\end{aligned}
$$

The first equality holds by Definition 6. The second equality holds by rewriting the minimum as a two-step minimization. The third equality holds by the induction hypothesis. The last equality holds by Definition 7. This concludes the proof. □

Proof of Lemma 8

Proof : This proof is by induction. (i) By definition of the value functions $V_0^\alpha(j) = 0$ for all $j \in \{0, 1, \dots, \alpha \cdot I_N\}$. Hence, $V_0^\alpha(j) \geq V_0^\alpha(j+1)$ for all $j \in \{0, 1, \dots, \alpha \cdot I_N - 1\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^\alpha(j) \geq V_t^\alpha(j+1)$ for all $j \in \{0, 1, \dots, \alpha \cdot I_N - 1\}$. Let $j \in \{0, 1, \dots, \alpha \cdot I_N - 1\}$. Now, observe that

$$
\begin{aligned}
V_{t+1}^\alpha(j) &= \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,j\}\}} \left\{ (\alpha - l)d_i + V_t^\alpha(j-l) \right\} \\
&\quad + \sum_{i\in N} \mu_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,\alpha\cdot I_N - j\}\}} \left\{ V_t^\alpha(j+l) \right\} \\
&\geq \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,j+1\}\}} \left\{ (\alpha - l)d_i + V_t^\alpha(j+1-l) \right\} \\
&\quad + \sum_{i\in N} \mu_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,\alpha\cdot I_N - j\}\}} \left\{ V_t^\alpha(j+l) \right\} \\
&\geq \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,j+1\}\}} \left\{ (\alpha - l)d_i + V_t^\alpha(j+1-l) \right\} \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha+1, \alpha \cdot I_N\}) \\
&\geq \sum_{i\in N} \lambda_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,j+1\}\}} \left\{ (\alpha - l)d_i + V_t^\alpha(j+1-l) \right\} \\
&\quad + \sum_{i\in N} \mu_i^* \min_{l\in\{0,1,\dots,\min\{\alpha,\alpha\cdot I_N - (j+1)\}\}} \left\{ V_t^\alpha(j+1+l) \right\}. \\
&= V_{t+1}^\alpha(j+1).
\end{aligned}
$$

The first inequality holds as $V_t^\alpha(j-l) \geq V_t^\alpha(j+1-l)$ for all $l \in \{0, 1, \dots, \min\{\alpha, j\}\}$ (by the induction hypothesis) and the fact that adding a (possible) term to a set from which its minimum is selected will not increase the minimum. The second inequality holds as $\min_{l\in\{0,1,\dots,\min\{\alpha,\alpha\cdot I_N - j\}\}} \left\{ V_t^\alpha(j+l) \right\} \geq V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}) \geq V_t^\alpha(\min\{j+1+\alpha, \alpha \cdot I_N\})$. Note that the inequalities are a direct consequence of the induction hypothesis, namely $V_t^\alpha(j) \geq V_t^\alpha(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 1\}$. The third inequality holds as adding (possible) terms to a set from which its minimum is selected will not increase the minimum.

(ii) First, the value function will be rewritten. From (i) of Lemma 8 it follows for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ that

$$\begin{aligned}
V_{t+1}^\alpha(j) = &\sum_{i \in N} \lambda_i^* \min_{l \in \{0,1,\ldots,\min\{\alpha,j\}\}} \Big\{ (\alpha - l)d_i + V_t^\alpha(j-l) \Big\} + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}) \\
&= \sum_{i \in N} \lambda_i^* \left[\min_{l \in \{0,1,\ldots,\min\{\alpha,j\}\}} \Big\{ (j-l)d_i + V_t^\alpha(j-l) \Big\} + (\alpha - j)d_i \right] \\
&\quad + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}) \\
&= \sum_{i \in N} \lambda_i^* \left[\min_{l \in \{\max\{0,j-\alpha\},\ldots,j\}} \Big\{ l d_i + V_t^\alpha(l) \Big\} + (\alpha - j)d_i \right] \\
&\quad + \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}),
\end{aligned}$$

where the second equality holds as $(a - l)d_i = (\alpha - j)d_i + (j - l)d_i$. The third equality holds by substituting $j - l$ into a single variable.

In addition, we define for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
V_t^{\alpha_1}(j) &= \sum_{i \in N} \lambda_i^* \left[\min_{l \in \{\max\{0,j-\alpha\},\ldots,j\}} \Big\{ l d_i + V_t^\alpha(l) \Big\} + (\alpha - j)d_i \right] \\
V_t^{\alpha_2}(j) &= \sum_{i \in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\})
\end{aligned}$$

Note that $V_t^\alpha(j) = V_t^{\alpha_1}(j) + V_t^{\alpha_2}(j)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$.

Now, we will prove by induction that $V_t^\alpha(j) + V_t^\alpha(j+2) \geq 2 \cdot V_t^\alpha(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ and all $t \in \mathbb{N} \cup \{0\}$. By definition of the value functions $V_0^\alpha(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$. Hence, $V_0^\alpha(j) + V_0^\alpha(j+2) \geq 2 \cdot V_0^\alpha(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^\alpha(j) + V_t^\alpha(j+2) \geq 2 \cdot V_t^\alpha(j+1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$.

We first focus on $V_t^{\alpha_2}(j)$ and thereafter focus on $V_t^{\alpha_1}(j)$.

Let $j \in \{0, 1, \ldots, \alpha \cdot (I_N - 1) - 2\}$. Now, observe that

$$\begin{aligned}
V^{\alpha_2}_{t+1}(j) + V^{\alpha_2}_{t+1}(j+2) &= \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}) \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+2+\alpha, \alpha \cdot I_N\}) \\
&= \sum_{i\in N} \mu_i^* V_t^\alpha(j+\alpha) + \sum_{i\in N} \mu_i^* V_t^\alpha(j+2+\alpha) \\
&= \sum_{i\in N} \mu_i^* \left(V_t^\alpha(j+\alpha) + V_t^\alpha(j+2+\alpha)\right) \\
&\geq 2\sum_{i\in N} \mu_i^* V_t^\alpha(j+1+\alpha) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+1+\alpha, \alpha \cdot I_N\}) \\
&= 2V^{\alpha_2}_{t+1}(j+1),
\end{aligned}$$

where the inequality holds by the induction hypothesis.

Let $j \in \{\alpha \cdot (I_N - 1) - 1, \alpha \cdot (I_N - 1), \ldots, \alpha \cdot I_N - 2\}$. Now, observe that

$$\begin{aligned}
V^{\alpha_2}_{t+1}(j) + V^{\alpha_2}_{t+1}(j+2) &= \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+\alpha, \alpha \cdot I_N\}) \\
&\quad + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+2+\alpha, \alpha \cdot I_N\}) \\
&\geq \sum_{i\in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) + \sum_{i\in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\alpha \cdot I_N) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{j+1+\alpha, \alpha \cdot I_N\}) \\
&= 2V^{\alpha_2}_{t+1}(j+1),
\end{aligned}$$

where the inequality holds by (i). Hence, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ it holds that $V^{\alpha_2}_{t+1}(j) + V^{\alpha_2}_{t+1}(j+2) \geq 2V^{\alpha_2}_{t+1}(j+1)$.

Let $j \in \{\alpha, \alpha+1, \ldots, \alpha \cdot I_N - 2\}$. Now, observe that

$$\begin{aligned}
&V^{\alpha_1}_{t+1}(j) + V^{\alpha_1}_{t+1}(j+2) \\
&= \sum_{i\in N} \lambda_i^* \min_{l \in \{\max\{0, j-\alpha\}, \ldots, j\}} \left\{ l d_i + V_t^\alpha(l) \right\} + \sum_{i\in N} \lambda_i^* \min_{l \in \{\max\{0, j+2-\alpha\}, \ldots, j+2\}} \left\{ l d_i + V_t^\alpha(l) \right\} \\
&= \sum_{i\in N} \lambda_i^* \min_{\substack{l_1 \in \{j-\alpha, \ldots, j\} \\ l_2 \in \{j+2-\alpha, \ldots, j+2\}}} \left\{ (l_1 + l_2) d_i + V_t^\alpha(l_1) + V_t^\alpha(l_2) \right\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i\in N}\lambda_i^* \min\Big[\{(2l_3+1)d_i + V_t^\alpha(l_3)+V_t^\alpha(l_3+1)|l_3 = j+1-\alpha,\ldots,j\}\\
&\qquad\qquad \cup\{2\left(V_t^\alpha(l_3)+l_3d_i\right)|l_3=j+1-\alpha,\ldots,j+1\}\Big]\\
&= 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{j+1-\alpha,j+2-\alpha,\ldots,j+1\}}\{V_t^\alpha(l_3)+l_3d_i\}\Big\}\\
&= 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{\max\{0,j+1-\alpha\},\ldots,j+1\}}\{V_t^\alpha(l_3)+l_3d_i\}\Big\}.\\
&= 2V_{t+1}^{\alpha_1}(j+1).
\end{aligned}
$$

The inequality holds as for any $l_1, l_2 \in \{0,1,\ldots,\alpha\cdot I_N - 2\}$ with $l_1 \leq l_2$ it holds that $V_t^\alpha(l_1)+V_t^\alpha(l_2) \geq V_t^\alpha(\lfloor(l_1+l_2)/2\rfloor)+V_t^\alpha(\lceil(l_1+l_2)/2\rceil)$ based on the induction hypothesis. This implies that for any $l_1, l_2 \in \{0,1,\ldots,\alpha\cdot I_N-2\}$ with l_1+l_2 odd, it follows that $(l_1+l_2)d_i + V_t^\alpha(l_1)+V_t^\alpha(l_2) \geq (2l_3+1)d_i + V_t^\alpha(l_3)+V_t^\alpha(l_3+1)$ where $l_3 = \lfloor(l_1+l_2)/2\rfloor$. For any $l_1,l_2\in\{0,1,\ldots,\alpha\cdot I_N-2\}$ with l_1+l_2 even, it follows that $(l_1+l_2)d_i+V_t^\alpha(l_1)+V_t^\alpha(l_2)\geq 2l_3d_i+2V_t^\alpha(l_3)$ with $l_3=(l_1+l_2)/2$. The third equality holds as $\min\{2a,a+b,2b\}=\min\{2a,2b\}$ for any $a,b\in\mathbb{R}$. The last but one equality holds as $j\geq\alpha$.

Let $j\in\{0,1,\ldots,\alpha-1\}$. Now, observe that

$$
\begin{aligned}
&V_{t+1}^{\alpha_1}(j)+V_{t+1}^{\alpha_1}(j+2)\\
&= \sum_{i\in N}\lambda_i^* \min_{l\in\{\max\{0,j-\alpha\},\ldots,j\}}\Big\{ld_i+V_t^\alpha(l)\Big\}+\sum_{i\in N}\lambda_i^* \min_{l\in\{\max\{0,j+2-\alpha\},\ldots,j+2\}}\Big\{ld_i+V_t^\alpha(l)\Big\}\\
&\geq \sum_{i\in N}\lambda_i^* \min_{\substack{l_1\in\{0,\ldots,j\}\\ l_2\in\{0,\ldots,j+2\}}}\Big\{(l_1+l_2)d_i+V_t^\alpha(l_1)+V_t^\alpha(l_2)\Big\}\\
&\geq \sum_{i\in N}\lambda_i^* \min\Big[\{(2l_3+1)d_i+V_t^\alpha(l_3)+V_t^\alpha(l_3+1)|l_3=0,\ldots,j\}\\
&\qquad\qquad \cup\{2\left(V_t^\alpha(l_3)+l_3d_i\right)|l_3=0,\ldots,j+1\}\Big]\\
&\qquad = 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{0,1,\ldots,j+1\}}\{V_t^\alpha(l_3)+l_3d_i\}\Big\}\\
&\qquad = 2\sum_{i\in N}\lambda_i^* \min_{l_3\in\{\max\{0,j+1-\alpha\},\ldots,j+1\}}\{V_t^\alpha(l_3)+l_3d_i\}\Big\}\\
&\qquad = 2V_{t+1}^{\alpha_1}(j+1).
\end{aligned}
$$

The first inequality holds as adding a (possible) term to a set from which its minimum is selected will not increase the minimum. The arguments of the other (in)equalities are similar to the ones of case $j \in \{\alpha, \alpha+1, \ldots, \alpha \cdot I_N - 2\}$. So, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$ it holds that $V^{\alpha_1}_{t+1}(j) + V^{\alpha_1}_{t+1}(j+2) \geq 2V^{\alpha_1}_{t+1}(j+1)$.

We conclude, for all $j \in \{0, 1, \ldots, \alpha \cdot I_N - 2\}$, it holds that

$$\begin{aligned} V^{\alpha}_{t+1}(j) + V^{\alpha}_{t+1}(j+2) &= V^{\alpha_1}_{t+1}(j) + V^{\alpha_2}_{t+1}(j) + V^{\alpha_1}_{t+1}(j+2) + V^{\alpha_2}_{t+1}(j+2) \\ &\geq 2V^{\alpha_1}_{t+1}(j+1) + 2V^{\alpha_2}_{t+1}(j+1) \\ &= 2V^{\alpha}_{t+1}(j+1). \end{aligned}$$

(iii) By definition of the value functions $V^{\alpha}_0(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$. Hence, $V^{\alpha}_0(k+j) + V^{\alpha}_0(k+j+2) = 2 \cdot V^{\alpha}_0(k+j+1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (I_N - 1)\alpha\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V^{\alpha}_t(k+j) + V^{\alpha}_t(k+j+2) = 2 \cdot V^{\alpha}_t(k+j+1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (I_N - 1)\alpha\}$.

First, observe that function $V^{\alpha}_t(j) + j \cdot d_i$ is convex in j for all $i \in N$ as $V^{\alpha}_t(\cdot)$ is convex by (ii) and $j \cdot d_i$ is linear. By our induction hypothesis, it holds that $V^{\alpha}_t(k+j) + V^{\alpha}_t(k+j+2) = 2 \cdot V^{\alpha}_t(k+j+1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha \ldots, (I_N - 1)\alpha\}$. As a consequence, it holds as well that $V^{\alpha}_t(k+j) + (k+j)d_i + V^{\alpha}_t(k+j+2) + (k+j+2)d_i = 2 \cdot V^{\alpha}_t(k+j+1) + 2 \cdot (k+j+1)d_i$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$, all $k \in \{0, \alpha, 2\alpha \ldots, \alpha(I_N - 1)\}$ and all $i \in N$. So, there exists an $h \in \{0, \alpha, 2\alpha, \ldots, \ldots, I_N\alpha\}$ for which it holds that $V^{\alpha}_t(h) + h \cdot d_i \leq V^{\alpha}_t(j) + j \cdot d_i$ for all $j \in \{0, 1, \ldots, \alpha \cdot I_N\}$ and all $i \in N$. Let $h \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$. Then, for all $k \in \{0, \alpha, \ldots, h - \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$\begin{aligned} &\sum_{i \in N} \lambda^*_i \left[\min_{l \in \{\min\{0, k+j-\alpha\}, \ldots, k+j\}} \left\{ ld_i + V^{\alpha}_t(l) \right\} + (\alpha - (k+j))d_i \right] \\ &= \sum_{i \in N} \lambda^*_i \left[(k+j)d_i + V^{\alpha}_t(k+j) + (\alpha - (k+j))d_i \right] \\ &= \sum_{i \in N} \lambda^*_i \left[V^{\alpha}_t(k+j) + \alpha \cdot d_i \right]. \end{aligned} \tag{4}$$

For $k = h$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$\begin{aligned} &\sum_{i \in N} \lambda^*_i \left[\min_{l \in \{\min\{0, k+j-\alpha\}, \ldots, k+j\}} \left\{ ld_i + V^{\alpha}_t(l) \right\} + (\alpha - (k+j))d_i \right] \\ &= \sum_{i \in N} \lambda^*_i \left[k \cdot d_i + V^{\alpha}_t(k) + (\alpha - (k+j))d_i \right] \\ &= \sum_{i \in N} \lambda^*_i \left[V^{\alpha}_t(k) + (\alpha - j) \cdot d_i \right]. \end{aligned} \tag{5}$$

For all $k \in \{h+\alpha, h+2\cdot\alpha, \dots, (I_N-1)\cdot\alpha\}$ and all $j \in \{0,1,\dots,\alpha\}$ it holds that

$$
\begin{aligned}
&\sum_{i\in N} \lambda_i^* \left[\min_{l\in\{\min\{0,k+j-\alpha\},\dots,k+j\}} \left\{ l d_i + V_t^\alpha(l) \right\} + (\alpha-(k+j))d_i \right] \\
&= \sum_{i\in N} \lambda_i^* \left[(k+j-\alpha)d_i + V_t^\alpha(k+j-\alpha) + (\alpha-(k+j))d_i \right] \\
&= \sum_{i\in N} \lambda_i^* \left[V_t^\alpha(k+j-\alpha) \right].
\end{aligned}
\tag{6}
$$

In addition, for all $k \in \{0, \alpha, \dots (I_N-2)\cdot\alpha\}$ and all $j \in \{0,1,\dots,\alpha-2\}$ it holds that

$$
\begin{aligned}
&\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+\alpha, \alpha\cdot I_N\}) + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+2+\alpha, \alpha\cdot I_N\}) \\
&= \sum_{i\in N} \mu_i^* V_t^\alpha(k+j+\alpha) + \sum_{i\in N} \mu_i^* V_t^\alpha(k+j+2+\alpha) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(k+j+1+\alpha) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+1+\alpha, \alpha\cdot I_N\})
\end{aligned}
$$

Moreover, for $k = (I_N-1)\cdot\alpha$ and all $j \in \{0,1,\dots,\alpha-2\}$ it holds that

$$
\begin{aligned}
&\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+\alpha, \alpha\cdot I_N\}) + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+2+\alpha, \alpha\cdot I_N\}) \\
&= \sum_{i\in N} \mu_i^* V_t^\alpha(\alpha\cdot I_N) + \sum_{i\in N} \mu_i^* V_t^\alpha(\alpha\cdot I_N) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\alpha\cdot I_N) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+1, \alpha\cdot I_N\})
\end{aligned}
$$

Hence, for all $k \in \{0, \alpha, \dots, (I_N-1)\cdot\alpha\}$ and all $j \in \{0,1,\dots,\alpha-2\}$ it holds that

$$
\begin{aligned}
&\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+\alpha, \alpha\cdot I_N\}) + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+2+\alpha, \alpha\cdot I_N\}) \\
&= 2\sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+1, \alpha\cdot I_N\}).
\end{aligned}
\tag{7}
$$

For all $k \in \{0,1,\dots,h-\alpha\}$ and all $j \in \{0,1,\dots,\alpha-2\}$ it holds that

$$
\begin{aligned}
&V_{t+1}^\alpha(k+j) + V_{t+1}^\alpha(k+j+2) \\
&= \sum_{i\in N} \lambda_i^* \left[V_t^\alpha(k+j) + \alpha\cdot d_i \right] + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+\alpha, \alpha\cdot I_N\}) \\
&\quad + \sum_{i\in N} \lambda_i^* \left[V_t^\alpha(k+j+2) + \alpha\cdot d_i \right] + \sum_{i\in N} \mu_i^* V_t^\alpha(\min\{k+j+2+\alpha, \alpha\cdot I_N\})\bigg) \\
&=2\sum_{i\in N} \lambda_i^* \left[V_t^\alpha(k+j+1) + \alpha\cdot d_i \right] + \sum_{i\in N} 2\mu_i^* V_t^\alpha(\min\{k+j+1+\alpha, \alpha\cdot I_N\}) \\
&=2V_{t+1}^\alpha(k+j+1).
\end{aligned}
$$

The first equality holds by (4). The second equality holds by the induction hypothesis and (7).

For $k = h$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$\begin{aligned}
&V_{t+1}^{\alpha}(k+j) + V_{t+1}^{\alpha}(k+j+2) \\
=&\sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k) + (\alpha - j) \cdot d_i \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{k+j+\alpha, \alpha \cdot I_N\}) \\
&+ \sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k) + (\alpha - (j+2)) \cdot d_i \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{k+j+2+\alpha, \alpha \cdot I_N\}) \\
=&2 \sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k) + (\alpha - (j+1)) \cdot d_i \right] + \sum_{i \in N} 2\mu_i^* V_t^{\alpha}(\min\{k+j+1+\alpha, \alpha \cdot I_N\}) \\
=&2V_{t+1}^{\alpha}(k+j+1).
\end{aligned}$$

The first equality holds by (5). The second equality holds by the induction hypothesis and (7).

For all $k \in \{h + \alpha, h + 2 \cdot \alpha, \ldots, (I_N - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$\begin{aligned}
&V_{t+1}^{\alpha}(k+j) + V_{t+1}^{\alpha}(k+j+2) \\
=&\sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k+j-\alpha) \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{k+j+\alpha, \alpha \cdot I_N\}) \\
&+ \sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k+j+2-\alpha) \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{k+j+2+\alpha, \alpha \cdot I_N\}) \\
=&2 \sum_{i \in N} \lambda_i^* \left[V_t^{\alpha}(k+j+1-\alpha) \right] + \sum_{i \in N} 2\mu_i^* V_t^{\alpha}(\min\{k+j+1+\alpha, \alpha \cdot I_N\}) \\
=&2V_{t+1}^{\alpha}(k+j+1).
\end{aligned}$$

The first equality holds by (6). The second equality holds by the induction hypothesis and (7). This concludes the proof. □

Proof of Lemma 9

Proof : Based on (i) and (iii) of Lemma 8, it follows directly that for all $j \in \{0, \alpha, \ldots, \alpha \cdot I_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that

$$V_{t+1}^{\alpha}(j) = \sum_{i \in N} \left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}} \{V_t^{\alpha}(j-l) + (\alpha - l) d_i\} \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j+\alpha, \alpha \cdot I_N\})$$

By definition of the value functions $V_0^{\alpha}(j) = 0$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$ and $V_0^N(j) = 0$ for all $j \in \{0, 1, \ldots, I_N\}$. Hence, $V_0^{\alpha}(j) = \alpha \cdot V_0^N\left(\frac{j}{\alpha}\right)$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^{\alpha}(j) = \alpha \cdot V_t^N\left(\frac{j}{\alpha}\right)$ for all $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$.

Let $j \in \{0, \alpha, \ldots, I_N \cdot \alpha\}$. Then, observe that

$$\begin{aligned}
V_{t+1}^{\alpha}(j) &= \left(\sum_{i \in N} \left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}} \{V_t^{\alpha}(j-l) + (\alpha - l) d_i\} \right] + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j+\alpha, \alpha \cdot I_N\} \right) \\
&= \left(\sum_{i \in N} \left[\lambda_i^* \min_{l \in \{0, \min\{j, \alpha\}\}} \left\{ \alpha \cdot V_t^N \left(\frac{j-l}{\alpha} \right) + (\alpha - l) d_i \right\} \right] \right. \\
&\qquad \left. + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j+\alpha, \alpha \cdot I_N\}) \right) \\
&= \left(\sum_{i \in N} \left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}} \left\{ \alpha \cdot V_t^N \left(\frac{j}{\alpha} - z \right) + \alpha \cdot (1 - z) d_i \right\} \right] \right. \\
&\qquad \left. + \sum_{i \in N} \mu_i^* V_t^{\alpha}(\min\{j+\alpha, \alpha \cdot I_N\} \right) \\
&= \left(\sum_{i \in N} \left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}} \left\{ \alpha \cdot V_t^N \left(\frac{j}{\alpha} - z \right) + \alpha \cdot (1 - z) d_i \right\} \right] \right. \\
&\qquad \left. + \sum_{i \in N} \alpha \cdot \mu_i^* V_t^N(\min\{\frac{j}{\alpha} + 1, I_N\} \right) \\
&= \alpha \cdot \left(\sum_{i \in N} \left[\lambda_i^* \min_{z \in \{0, \min\{\frac{j}{\alpha}, 1\}\}} \left\{ V_t^N \left(\frac{j}{\alpha} - z \right) + (1 - z) d_i \right\} \right] \right. \\
&\qquad \left. + \sum_{i \in N} \mu_i^* V_t^N(\min\{\frac{j}{\alpha} + 1, I_N\} \right) \\
&= \alpha \cdot V_{t+1}^N \left(\frac{j}{\alpha} \right).
\end{aligned}$$

The first equality holds by definition. The second equality holds by the induction hypothesis. The third equality holds by introducing a new variable $z = l/\alpha$. The fourth equality holds by the induction hypothesis (again). The fifth equality holds as α can be taken outside the summations. The last equality holds by Lemma 1. $\square$