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Spare parts pooling games under a critical level policy

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Abstract

We consider an environment in which several independent service providers with (possibly) criticality differences can reduce their joint downtime costs by pooling their spare parts according to a fixed critical level policy. We examine the allocation of the collective cost savings for such a pooled situation by studying an associated cooperative game. For this game, which we call a criticality game, we will first derive necessary and sufficient conditions for convexity. Thereafter, we will show that the values of criticality games may be recognized as convex combinations of values of Böhm–Bahwerk Horse Market games. As a consequence, we can show that criticality games have non–empty cores. In addition, we will show that criticality games are totally balanced and present a class of allocation rules for which the resulting payoff vectors are core members. Finally, we study a simple and intuitive allocation rule within this class of allocation rules.

Keywords : cooperative game; spare parts pooling; critical level policy; downtime cost savings; allocation rules

1 Introduction

The Netherlands has one of the most intensively used railway networks of the world. Every day, more than one million passengers are transported on a limited railway network of 6830 kilometers length only (Ramaekers et al. [15]). A highly reliable and available railway is needed in order to guarantee such kind of service. However, disruptions may occur, which can affect the availability of the railway network negatively. For that reason, disruptions should be repaired quickly. This requires bringing the right service engineers, the right equipment and the right spare parts to the disruption. The current administrator of the Dutch railway network has outsourced such (daily) maintenance activities to four competitive contractors (Schlicher et al. [17]). In practice, this means that each contractor maintains a part of the network and that spare parts control, which consists of a natural trade–off between holding costs and downtime costs, is organized locally. It is well–known that such decentralized spare parts control is suboptimal in terms of long–term average costs (see e.g. Paterson et al. [13]). For that reason, it could be of interest to centralize spare parts control, i.e., to pool spare parts. For such spare parts pooling collaboration, some interesting questions arise. The first one relates to which inventory pooling strategy to choose, especially in the light of criticality differences of railway track components between regions. For example, a switch located at the main railway station of the Netherlands is more critical than a switch located at the border of the network. The second one relates to the cooperation between the four (competitive) contractors. How should potential costs savings be allocated such that all parties are willing to cooperate?

When we restrict ourselves by considering a single spare part and assume that an emergency procedure (for a spare part) is instigated when demand cannot be satisfied immediately (which is typically the case in the Dutch railway setting), the Dutch railway situation fits within the framework of a single–item inventory model with multiple demand classes and lost sales (i.e., no backorders) where holding costs are paid over spare parts on stock and class specific penalty costs (i.e., service provider–specific downtime costs) are incurred when demand cannot be satisfied. For such an environment, so–called *critical level policies* can be used to reduce long-term average costs significantly (see e.g. Dekker et al. [3]). In particular, when leadtimes are exponentially distributed, one can proof optimality of a critical level policy (Ha [6]). Under a critical level policy, a service provider is allowed to use spare parts when inventory on–hand is above a specific critical level only. In this way, some spare parts are reserved for the more critical service providers, e.g., for contractors with relatively high downtime costs. Although critical level policies have been investigated intensively for several underlying settings (see, e.g., Deshpande et al. [4], Kleijn and Dekker [9], Kranenburg and Van Houtum [10]), the combination with cost allocation has only received attention for the specific setting wherein *full pooling*, which is a particular critical level policy, is applied.

Wong et al. [24] were the first who used game theoretical models to investigate the cost allocation aspect in the context of repairable spare parts pooling. They proposed four cost allocation policies and analysed them in a numerical experiment. In Karsten et al. [8] and Karsten and Basten [7] a setting is investigated with several independent decision makers that all stock expensive, low–demand spare parts for their high–tech machines. Those decision makers can collaborate by full pooling of their spare parts. Karsten et al. [8] and Karsten and Basten [7] examined the stability of such pooling arrangements and showed (under certain intuitive conditions) the existence of stable allocations. Recently, Guajardo and Rönnqvist [5] investigated a setting with several decision makers having different target service levels for spare parts with backordering. When collaborating, new target service levels are set and full pooling is assumed. Again, concepts from cooperative game theory are used to analyse the cost allocation aspect.

As already mentioned, a major overlap in all above papers is the underlying assumption of full pooling. From Dekker et al. [3] and Ha [6] we know that full pooling may be far from optimal in situations with significant criticality differences, i.e., with significant downtime costs differences. One could improve system performance if another critical level policy is applied. We will analyse such a critical level policy, which may be suboptimal as well, but can outperform full pooling, e.g., when downtime costs differ significantly. To the best of our knowledge, there exists no literature that investigates a cost allocation aspect in combination with spare parts pooling according to a critical level policy, other than full pooling. In this paper, we make a first step in this direction.

We consider an environment in which several service providers each keep a *single* stock–keeping unit for a single type of high–tech machine or equipment, e.g., a track segment of the railway network. Unit holding costs are assumed to be equal for all service providers, while unit downtime costs may differ between service providers. We assume that spare parts can be transshipped instantaneously at negligible costs. In many settings, this assumption is reasonable, as (*i*) removing a defect component of a high–tech machine (or equipment) may already take hours, which makes that transhipment will not affect the duration of the replacement and (*ii*) transshipment costs are often negligible compared to (high) downtime costs. In addition, we assume that the intensity by which failures occur is equal for all the service providers. This assumption is reasonable in situations where service providers have similar amounts of high-tech machines (or equipment) that are used frequently, e.g., in the Dutch railway network several regions have similar amounts of switches. Service providers can cooperate by pooling their spare parts according to a fixed critical level policy. Under this policy the number of available spare parts left determines which service providers are allowed to satisfy demand. If one spare part is left, the service provider with the highest downtime costs is allowed to satisfy demand only. When two spare parts are left, the service providers with the highest and second highest downtime costs are allowed to satisfy demand only, and so on. As one–by–one players are added to the group of players that are allowed to satisfy demand, we refer to this critical level policy as the *one–by–one* critical level policy. In our situation, the one–by–one critical level policy has the appealing property that, under full cooperation, the long-term average costs are at most equal to the sum of the long-term average costs when the service providers operate individually.

We will examine the allocation of collective costs savings for such a pooled situation by studying an associated cooperative game. For this cooperative game, which we call a *criticality game*, we will present necessary and sufficient conditions to ensure that each player's marginal contribution increases as the coalition to which he belongs grows larger, i.e., we will show necessary and sufficient conditions to ensure convexity of this criticality game. In addition we will show that the values of any criticality game may be recognized as a convex combination of values of Böhm–Bawerk Horse Market games (see e.g. Núñez and Rafels [12]). As a consequence, we can show our main contribution: for every criticality game there exists at least one efficient payoff vector that cannot be improved upon by any coalition, i.e., the core of this game is non–empty. It shows that, under the one–by–one critical level policy, there always exists an incentive for service providers to cooperate! In addition, we will show that every subgame of a criticality game has a non-empty core as well, i.e., criticality games are totally balanced. Moreover, we present a class of allocation rules for which the resulting payoff vectors are core members. Finally, we show that there exists an intuitive and simple allocation rule in this class of allocation rules. The resulting payoff vectors of this allocation rule coincide with the average of so called extreme BBHM payoff vectors. An extreme BBHM payoff vector is defined as a payoff vector resulting from an allocation rule of the class of allocation rules, where based on an arbitrary order of the player set, a player selects core allocations in its own best interest of BBHM games in which he is active, under the restriction that the players before him in the corresponding order recursively selected their related core allocations. Although our focus is spare parts oriented, our results may be applicable to other resource systems as well (see e.g. Anily and Haviv [1] or Reinhardt and Dada [16]). For instance, it may be applied to individual taxi drivers with different contractual agreements for similar-sized companies. These taxi drivers may pool their taxis to increase total profit.

The remainder of this paper is as follows. We start in Section 2 with preliminaries on cooperative game theory. In Section 3 criticality situations and corresponding criticality games are introduced. In Section 4, we analyse criticality games. Finally, conclusions will be drawn in Section 5.

2 Preliminaries on cooperative game theory

In this section, we provide some basic elements of cooperative game theory. Consider a finite set of *players* $N \subset \mathbb{N}$ and a function $v : 2^N \to \mathbb{R}$ called a *characteristic function*, with $v(\emptyset) = 0$. The pair (N, v) is called a *cooperative game with transferable utility*, shortly called *game*. A subset $M \subseteq N$ is a *coalition* and

$v(M)$ is the worth that coalition M can achieve by itself. The worth can be transferred freely among the players. The set N is called the *grand coalition*. For a given coalition $M \subseteq N$, the *subgame* (M, v_M) is the game with player set M and characteristic function v_M such that $v_M(T) = v(T)$ for all $T \subseteq M$. A game (N, v) is called *superadditive* if the value of the union of two disjoint coalitions is more than or equal to the sum of the values of the disjoint coalitions, i.e., $v(M) + v(T) \leq v(M \cup T)$ for all $M, T \subseteq N$ with $M \cap T = \emptyset$. A game (N, v) is called *monotonic* if the value of every coalition is at least the value of any of its subcoalitions, i.e., $v(M) \leq v(T)$ for all $M, T \subseteq N$ with $M \subseteq T$ and *convex* if the marginal contribution of any player to any coalition is less than his marginal contribution to a larger coalition, i.e., $v(M \cup \{i\}) - v(M) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $M \subset T \subseteq N \backslash \{i\}$. A *payoff vector* for a game (N, v) is a vector $x \in \mathbb{R}^N$ describing the payoffs to the players, where player $i \in N$ receives x_i. A payoff vector is called *efficient* if $\sum_{i \in N} x_i = v(N)$. This implies that all worth is divided among the players of the grand coalition N. A payoff vector is called *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$ and called *stable* if no group of players has an incentive to leave the grand coalition N, i.e., $\sum_{i \in M} x_i \geq v(M)$ for all $M \subseteq N$. The set of efficient and individually rational payoff vectors of (N, v), called the *imputation set* of (N, v), is denoted by $I(N, v)$. The set of efficient and stable payoff vectors of (N, v), called the *core* of (N, v), is denoted by $C(N, v)$. Following Bondareva [2] and Shapley [19], a game (N, v) is called *balanced* if the core is non–empty. If for every $M \subseteq N$, the corresponding subgame (M, v_M) is balanced, the game is called *totally balanced*.

A Böhm–Bawerk Horse Market (BBHM) situation is a two–sided market with homogenous goods, e.g., horses. In this market, there are sellers that each have one good for sale and buyers that each want to buy one good. The finite set of sellers and buyers together is denoted by $N \subset \mathbb{N}$. The set of sellers is denoted by $S \subseteq N$ and the set of buyers is denoted by $B = N \backslash S$. Each seller (or buyer) $i \in N$ has a valuation w_i for its good. Without loss of generality, we assume that $w_i \geq w_j$ if $i, j \in N$ with $i < j$.

Definition 1. *A Böhm–Bawerk Horse market situation is a tuple* (N, S, w) *with*

- $N \subset \mathbb{N}$ *the finite set of sellers and buyers;*

- *$S \subseteq N$ the set of sellers (and $N \backslash S (= B)$ the set of buyers);*
- *$w = (w_i)_{i \in N}$ is the vector of valuations, where $w_i \geq w_j$ if $i, j \in N$ with $i < j$.*

We denote the set of BBHM situations by Γ. For every $\gamma \in \Gamma$, it holds for any buyer $j \in B$ and any seller $i \in S$ that if $w_j < w_i$, no good will be traded between buyer j and seller i, and if $w_j \geq w_i$, buyer j and seller i can trade a good with a joint profit of $w_j - w_i \geq 0$. Now, let $A_{S,B}$ be defined as $A_{S,B} = (a_{ij})_{i \in S, j \in B}$, where $a_{ij} = \max\{w_j - w_i, 0\}$. A matching between set $S_1 \subseteq S$ and set $B_1 \subseteq B$ is a subset of pairs $\mu \subseteq S_1 \times B_1$, where each seller (of S_1) and each buyer (of B_1) belongs to at most one pair in μ. Let $\mathscr{M}(S_1, B_1)$ be the set of all such possible matchings. A matching $\mu \in \mathscr{M}(B_1, S_1)$ is called *optimal* on (B_1, S_1) if for all $\mu' \in \mathscr{M}(B_1, S_1)$ it holds that

$$\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij}.$$

Note that an optimal matching on (B_1, S_1) always exists as the total number of possible matchings is finite. The joint profit of any coalition $T \subseteq N$ is defined as the sum of the joint profits of all pairs of an optimal matching $\mu \in \mathscr{M}(S \cap T, B \cap T)$. Note that if T exists of buyers only or sellers only, then this joint profit equals zero. Now, we will associate a BBHM game to any BBHM situation.

Definition 2. *For any $\gamma = (N, S, w) \in \Gamma$, the corresponding BBHM game (N, v^γ) is given by*

$$v^\gamma(T) = \max_{\mu \in \mathscr{M}(S \cap T, B \cap T)} \left\{ \sum_{(i,j) \in \mu} a_{ij} \right\} \quad \text{for all } T \subseteq N.$$

BBHM games are well–studied (see e.g. Shapley and Shubik [21], Moulin [11] and Núñez and Rafels [12]). In particular, Shapley and Shubik [21] showed that the core of a BBHM game consists of a line segment. In order to define this segment, we introduce some additional notation. For any $M \subseteq N$, we define bijection $\sigma_M : M \to \{1, 2, \ldots, |M|\}$ with $\sigma_M(i) = |\{1, 2, \ldots, i\} \cap M|$ for any $i \in M$, i.e., $\sigma_M(i)$ is the position of player $i \in M$ in coalition M. Moreover, let function σ_M^{-1} be defined as the inverse of σ_M. Let $z = \min\{|S|, |B|\}$ and $l = \max\{i \in \{1, 2, ..., z\} \mid w_{\sigma_B^{-1}(i)} - w_{\sigma_S^{-1}(|S|+1-i)} \geq 0\}$, i.e., l indicates the number of sellers (or buyers) that trade a good. Those sellers (buyers) are called *active* sellers (buyers) and the remaining sellers (buyers) are called *non-active* sellers (buyers). For any BBHM situation $(N, S, w) \in \Gamma$ with $S \neq \emptyset, N$, i.e., with at least one and at most $|N| - 1$ sellers, we introduce

$$\underline{\delta} = \max\left\{w_{\sigma_S^{-1}(|S|-l+1)}\ ,\ w_{\sigma_B^{-1}(l+1)}\right\}\quad,\quad \overline{\delta}\ = \min\left\{w_{\sigma_S^{-1}(|S|-l)}\ ,\ w_{\sigma_B^{-1}(l)}\right\},$$

with $w_{\sigma_S^{-1}(0)} = \infty$ and $w_{\sigma_B^{-1}(|B|+1)} = -\infty$. For any BBHM situation $\gamma \in \Gamma$ with $S = \emptyset$, let $\underline{\delta} = \overline{\delta} = w_1$ and for any BBHM situation $\gamma \in \Gamma$ with $S = N$, let $\underline{\delta} = \overline{\delta} = w_n$. We will refer to $\underline{\delta}$ and $\overline{\delta}$ as the extreme *prices* to trade a good. Now, for any $\gamma \in \Gamma$, vector $\mathscr{L}(\gamma, \alpha)$ defined by

$$\mathscr{L}_i(\gamma, \alpha) = \begin{cases} \max\{(\underline{\delta} + \alpha((\overline{\delta} - \underline{\delta}))) - w_i, 0\} & \text{if } i \in S \\ \\ \max\left\{w_i - (\underline{\delta} + \alpha((\overline{\delta} - \underline{\delta}))), 0\right\} & \text{if } i \in B \end{cases}$$

with $\alpha \in [0, 1]$ for all $i \in N$, is a core element of the corresponding BBHM game. Varying between $\alpha = 0$, i.e., the *buyer optimal* core allocation, and $\alpha = 1$, i.e., the *seller optimal* core allocation, describes the whole core segment.

Theorem 1 (Shapley and Shubik [21])**.** *For every $\gamma \in \Gamma$ the core of the corresponding BBHM game (N, v^γ) is non-empty. In particular, the core is given by*

$$C(N, v^\gamma) = \left\{\mathscr{L}(\gamma, \alpha) \in \mathbb{R}^N_+ \ \middle|\ \alpha \in [0, 1]\right\}.$$

One can check easily that for any *non–active* player $i \in N$, i.e., any player that will not trade a good, $\mathscr{L}_i(\gamma, \alpha) = 0$. We end this section with an example.

Example 1. *Consider BBHM situation $\gamma = (N, s, w) \in \Gamma$ with $N = \{1, 2, 3\}$, $S = \{2\}$ and $w = (5, 2, 1)$. Then, $\underline{\delta} = \max\{2, 1\} = 2$ and $\overline{\delta} = \min\{\infty, 5\} = 5$. As a consequence $C(N, v^\gamma) = \{(3 \cdot (1 - \alpha), 3 \cdot \alpha, 0) \mid \alpha \in [0, 1]\}$.* ⋄

3 Model

In this section, we introduce criticality situations and define the associated games, called criticality games.

3.1 Criticality situation

Consider a service provider i who keeps spare parts on stock to prevent costly downtime of its machines. We limit ourselves to one critical component, i.e., one stock–keeping unit, which is subject to failures. A failure of a machine immediately leads to a demand for a spare part. This occurs according to a Poisson process with stationary rate $\lambda \in \mathbb{R}_+$. We assume that service provider i starts with *one* spare part in stock only. If a spare part is on hand when demand occurs, this demand is always satisfied and a replenishment order is placed immediately. The lead times of such orders are independent and identically distributed according to a general distribution function with mean $\mu^{-1} \in \mathbb{R}_+$. If no spare part is available when demand occurs, an external spare part is ordered immediately and the machine goes down until the failed component is replenished by the external spare part. The expected costs associated with the extra idleness of the machine, shipments, and so on, shortly called downtime costs, are $d_i \in \mathbb{R}_+$ for service provider i. The underlying (spare parts) policy can be recognised as a continuous–review base stock policy with base stock level one. The holding costs per spare part are $h \in \mathbb{R}_+$ per time unit and are paid over inventory on-hand only. Finally, we assume that the service provider is interested in its long-term average costs per time unit.

Now, consider an environment with several such service providers. We formalize this situation by a tuple, which we refer to as a criticality situation.

Definition 3. *A criticality situation is a tuple (N, μ, λ, d, h), where*

- $N \subset \mathbb{N}$ *is the finite set of players (a player corresponds to a service provider);*
- $\mu \in \mathbb{R}_+$ *the inverse of the mean of a general distribution function generating lead times of replenishment orders for all players;*
- $\lambda \in \mathbb{R}_+$ *the demand rate of the Poisson process (which is the same) for each player;*
- $d = (d_i)_{i \in N} \in \mathbb{R}_+^N$ *is the vector of downtime costs, where $d_i \leq d_j$ if $i, j \in N$ and $i < j$ (d_i is the downtime cost of player i);*
- $h \in \mathbb{R}_+$ *is the holding cost per spare part per unit time for each player.*

For short, we use θ to refer to a criticality situation $\theta = (N, \mu, \lambda, d, h)$ and Θ for the set of all criticality situations. As the names of players can be relabeled, the non-increasing property of the vector of downtime costs is without loss of generality.

3.2 Criticality game

Players can cooperate by pooling their spare parts. For any coalition $M \subseteq N$ we assume that (*i*) transshipments (of spare parts) occur instantaneously at negligible costs and (*ii*) the one–by–one critical level policy is applied. Under this policy, demand of player $i \in M$ is filled as long as inventory on–hand is at least equal to critical level $\sigma_M(i)$[1]. This implies that the player with the highest downtime costs can satisfy demand as long as inventory on–hand is at least one, the player with the second highest downtime costs can satisfy demand as long as inventory on–hand is at least two, and so on. In addition, after any satisfied demand, a replenishment order is placed immediately. If player $i \in M$ faces demand, while inventory on–hand is below critical level $\sigma_i(M)$, an external spare part is used and related downtime costs of d_i are incurred. Note that for coalitions with cardinality one, a one–by–one critical level policy reduces to a continuous–review base stock level policy with base stock level one. As players are interested in the long–term average costs per time unit, we will determine the steady state probabilities of pooling group M with $m = |M|$ players and $i \in \{0, 1, 2, \ldots, m\}$ spare part(s) on stock. Let $\pi(m, i)$ be defined as the steady state probability of coalition M with m players and i spare part(s) on stock. The following lemma presents a (well–known) closed–form description of these steady state probabilities.

Lemma 1. *For every criticality situation $\theta \in \Theta$ the steady state probabilities are given for all $M \subseteq N$ and all $i \in \{0, 1, \ldots, m\}$ by*

$$\pi(m, i) = \binom{m}{i} \cdot \left(\frac{\mu}{\lambda + \mu}\right)^{i} \cdot \left(\frac{\lambda}{\lambda + \mu}\right)^{m-i}.$$

Proof : See e.g. Van Houtum and Kranenburg [23, p.74, eq. 4.1].

For coalition $M \subseteq N$ the expected costs per time unit in state i are $\pi(m, i) \cdot i \cdot h$ for holding spare parts and $\pi(m, i) \cdot \lambda \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}$ for the related downtime costs.

[1] Recall $\sigma_M(i) = |\{1, 2, \ldots, i\} \cap M|$.

Summing over all $m+1$ states results in the expected costs per time unit in steady state for coalition $M \subseteq N$ under criticality situation $\theta \in \Theta$.

Definition 4. *For every criticality situation $\theta \in \Theta$ the expected costs per time unit in steady state for any coalition $M \subseteq N$ are denoted by*

$$c^{\theta}(M) = \sum_{i=0}^{m} \left[\pi(m,i) \cdot \left(i \cdot h + \lambda \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right]. \tag{1}$$

Example 2. *Consider criticality situation $\theta = (N, \mu, \lambda, d, h) \in \Theta$ with $N = \{1,2,3\}$, $d_1 = 5$, $d_2 = 2, d_3 = 1$, $\lambda = 1, \mu = 2$, and $h = \frac{1}{2}$. In Table 1, the steady state probabilities for every possible cardinality of coalitions are presented.*

Table 1: Steady state probabilities

i	0	1	2	3
$\pi(1,i)$	$\frac{1}{3}$	$\frac{2}{3}$		
$\pi(2,i)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	
$\pi(3,i)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

For example, for coalition $M = \{1,2\}$ we obtain

$$c^{\theta}(\{1,2\}) = \frac{1}{9} \cdot (1 \cdot (5+2)) + \frac{4}{9} \cdot \left(1 \cdot \frac{1}{2} + 1 \cdot 2\right) + \frac{4}{9} \cdot \left(2 \cdot \frac{1}{2}\right) = 2\frac{1}{3}.$$

Similarly, the costs of the other coalitions can be determined. In Table 2, the corresponding costs for all coalitions are depicted. ◇

Table 2: Corresponding costs per coalition

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c^{\theta}(M)$	0	2	1	$\frac{2}{3}$	$2\frac{1}{3}$	$1\frac{7}{9}$	$1\frac{4}{9}$	$2\frac{11}{27}$

Note that the holding costs in Example 2 are additive, i.e., the sum of the holding costs of any group of individual players equals the holding costs of those players when cooperating. This turns out not to be a coincidence.

Lemma 2. *For every criticality situation $\theta \in \Theta$, holdings costs are additive, i.e.,*

$$m \cdot (h \cdot \pi(1,1)) = h \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i \quad \text{for all } M \subseteq N$$

Proof : See Appendix.

The result of Lemma 2 may be counterintuitive as pooling of spare parts normally leads to less inventory on–hand. However, due to the specific one-by-one critical level policy, it holds for any coalition $M \subseteq N$ that the relevant demand rate with $j \in N$ spare parts on stock is $j \cdot \lambda$ for both the situation with and the situation without cooperation. As a consequence, holding costs are additive.

In what follows, we formulate a criticality (cost saving) game corresponding to a criticality situation $\theta \in \Theta$.

Definition 5. *For any criticality situation $\theta \in \Theta$, the corresponding criticality (cost saving) game (N, v^θ) is defined by*

$$v^\theta(M) = \sum_{i \in M} c^\theta(\{i\}) - c^\theta(M) \tag{2}$$

for all coalitions $M \subseteq N$.

Based on Lemma 2, we may conclude that the cost *savings* do not depend on the holding costs. This implies that criticality games are completely determined by the savings in downtime costs. Note that for criticality situations with other additive holding costs structures, e.g., $|M| \cdot h$ for any coalition $M \subseteq N$, we would end up with the same criticality game.

Lemma 3. *For every criticality situation $\theta \in \Theta$, it holds for all $M \subseteq N$ that*

$$v^\theta(M) = \lambda \cdot \sum_{i=1}^{m-1} \left[\pi(m,i) \cdot \left(\frac{m-i}{m} \cdot \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right]. \tag{3}$$

Proof : See Appendix.

Observe that the term between the inner brackets of (3) can be interpreted as the difference between the cost savings at the first i players and the extra costs at the remaining $m-i$ players.

Using Lemma 3, we derive that players are at least as well off in a coalition rather than working individually. Equivalently, the long–term average costs under the one–by–one critical level policy are at most equal to the sum of the long–term average costs when service providers work individually.

Lemma 4. *For every criticality situation $\theta \in \Theta$ and all $M \subseteq N$ it holds that*

$$v^{\theta}(M) \geq 0.$$

Proof : See Appendix.

We conclude this section with an example of a criticality game.

Example 3. *Consider the situation of Example 2. The cost savings for every coalition of the criticality game can be calculated directly via Lemma 3, or indirectly via Definition 5 if the corresponding criticality game is already determined. For example, the value of coalition $M = \{1,2\}$ is*

$$v^{\theta}(\{1,2\}) = c^{\theta}(\{1\}) + c^{\theta}(\{2\}) - c^{\theta}(\{1,2\}) = 2 + 1 - 2\frac{1}{3} = \frac{2}{3}.$$

In a similar way, all other values can be determined. The cost savings for every coalition of the criticality game are presented in Table 3. ◇

Table 3: Corresponding costs savings per coalition

M	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{\theta}(M)$	0	0	0	0	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{2}{9}$	$1\frac{7}{27}$

4 Game properties and profit allocations

In this section, we first investigate whether criticality games satisfy interesting properties. We start by considering convexity and then investigate whether the cores of criticality games are non empty. Thereafter, we introduce and analyse a class of allocation rules. Finally, we study a simple and intuitive allocation rule within this class of allocation rules.

4.1 Convexity

The Shapley value is a core member when the corresponding game is convex (Shapley [20]). As a consequence, the core of a convex game is non–empty. For that reason, we first investigate whether criticality games are convex in general. The following example illustrates that this is not the case.

Example 4. *Consider the situation of Example 3. Observe that* $v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,3\}) = 1\frac{7}{27} - \frac{8}{9} = \frac{10}{27} < \frac{18}{27} = \frac{2}{3} - 0 = v^{\theta}(\{1,2\}) - v^{\theta}(\{1\})$ *and we can conclude that the game is not convex.* ◇

Despite that criticality games are not convex in general, we can identify necessary and sufficient conditions for which convexity can be ensured.

Theorem 2. *For every criticality situation* $\theta \in \Theta$ *with* $|N| \geq 3$ *the corresponding criticality game is convex if and only if*

$$d_i - 2d_j + d_k = 0 \quad \forall i, j, k \in N \text{ with } i < j < k \tag{4}$$

Proof : See Appendix.

Note that for criticality situations with three players only, the corresponding criticality game is convex if and only if $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(3)} = 0$. For criticality situations with more than three players, the game is convex if and only if all downtime costs are equal implying that the game is additive.

Corollary 1. *For every criticality situation* $\theta \in \Theta$ *with* $|N| \geq 4$ *the criticality game is convex if and only if there exists a* $z \in \mathbb{R}_+$ *such that* $d_i = z \in \mathbb{R}_+$ *for all* $i \in N$.

4.2 The core

The results of Section 4.1 provide a necessary and sufficient condition for convexity of criticality games. These results leave the issue of core non-emptiness unanswered for a large class of criticality games. This issue is the main topic of this section. The following example illustrates a phenomenon which turns out to be useful in here.

Example 5. *Consider a criticality situation $\theta \in \Theta$ with $N = \{1,2,3\}$. Suppose that player 1 is out of stock, while player 2 and player 3 do have a spare part available. Then, in the grand coalition, player 1 has the right to use the spare part of player 3 if necessary, based on the underlying one–by–one critical level policy, while plater 3 is not allowed to use its own spare part. Player 1 can be recognized as a buyer with value $\lambda \cdot d_1$ per time unit for a spare part and player 3 can be recognized as a seller with value $\lambda \cdot d_3$ per time unit for a spare part. So, an additional worth of $\lambda(d_1 - d_3) > 0$ per time unit can be realized. By Lemma 3 this situation has steady state probability $\frac{\lambda\mu^2}{(\lambda+\mu)^3}$. In a similar way, one can analyse the situations with player 2 out of stock, player 3 out of stock, player 1 and 2 out of stock, and so on. By combining all these additional worths and corresponding probabilities, we obtain $v^\theta(N)$.*

Example 5 illustrates that the value of the grand coalition of a 3–person game can in fact be recognized as a convex combination of values of BBHM games. We will formalize and prove this result in general. Let n be the cardinality of player set N, i.e., $n = |N|$. For every criticality situation $\theta \in \Theta$ we introduce 2^n different BBHM situations, where every BBHM situation has a unique set of sellers.

Let $\theta = (N, \mu, \lambda, d, h) \in \Theta$ be a criticality situation. Then, for every $S \subseteq N$ the related BBHM situation will be denoted by $\gamma_S^\theta = (N, S, w)$, where $w = (\lambda \cdot d_i)_{i \in N}$.

Example 6. *Consider the situation of Example 2. The games associated with all corresponding BBHM situations are depicted in Table 4. Note that $i \in \{1,2,3\}$.*

Table 4: Corresponding values for BBHM games

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{\gamma_S^\theta}(\{1,2,3\})$	0	0	$5-2$	$5-1$	0	$2-1$	$5-1$	0
$v^{\gamma_S^\theta}(\{1,2\})$	0	0	$5-2$	0	0	0	$5-2$	0
$v^{\gamma_S^\theta}(\{1,3\})$	0	0	0	$5-1$	0	0	$5-1$	0
$v^{\gamma_S^\theta}(\{2,3\})$	0	0	0	$2-1$	0	$2-1$	0	0
$v^{\gamma_S^\theta}(\{i\})$	0	0	0	0	0	0	0	0
$v^{\gamma_S^\theta}(\emptyset)$	0	0	0	0	0	0	0	0

For example, for coalition $M = \{1,3\}$, in BBHM situation $\gamma^{\theta}_{\{3\}}$, we have $\mathscr{M}(A_{S\cap M,B\cap M}) = \{\emptyset, \{(3,1)\}\}$. Hence, the value of coalition M is

$$v^{\gamma^{\theta}_{\{3\}}}(\{1,3\}) = \max_{\mu\in\{\emptyset,\{(3,1)\}\}} \left\{ \sum_{(i,j)\in\mu} a_{ij} \right\} = a_{31} = \max\{\lambda\cdot(d_3-d_1),0\} = 4. \qquad \diamond$$

Recall that for any BBHM situation γ^{θ}_{S} with $S \subseteq N$ a buyer (or seller) that buys (or sells) a spare part is called an active buyer (active seller), whether a buyer (seller) that buys (sells) no spare part is called a non–active buyer (non–active seller). As $d_i \geq d_j$ if $i,j \in N$ with $i < j$ we have $\lambda \cdot d_i \geq \lambda \cdot d_j$ if $i,j \in N$ with $i < j$. This implies the following result. Note that we use $s = |S|$ for every $S \subseteq N$.

Lemma 5. *For every criticality situation $\theta \in \Theta$ and every seller set $S \subseteq N$ there exists an optimal matching in γ^{θ}_{S} for which all players $i \in N$ with $1 \leq \sigma_N^{-1}(i) \leq s$ have (the right to use) a spare part after that specific optimal matching.*

In the remainder of this paper, we assume that the specific matching of Lemma 5 will be applied. Based on this assumption, it is straightforward to determine which buyers and sellers are the active and non–active ones. If player $i \in N$ is a seller, it is active if and only if $s+1 \leq \sigma_N^{-1}(i) \leq n$ and if player $i \in N$ is a buyer, it is active if and only if $1 \leq \sigma_N^{-1}(i) \leq s$.

Now, it is easy to determine (i) the coalitions in which a player is active (non–active) and subsequently (ii) the total joint profit under a given criticality situation $\theta \in \Theta$ and a fixed number of sellers.

Lemma 6. *For every $\theta \in \Theta$ it holds for all $k \in \mathbb{N}$ with $1 \leq k \leq n-1$ that*

$$\sum_{S\subseteq N:|S|=k} v^{\gamma^{\theta}_{S}}(N) = \lambda \cdot \binom{n}{k} \left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right). \tag{5}$$

Proof : See appendix.

Lemma 6 states that the total joint profit under a fixed number of sellers consists of the product of (i) the total number of possible sets with k sellers and (ii) the total value for the first k players to have the right to use spare parts minus the initial value of any player before interchanging, which occurs with chance k/n.

In order to state and show our main result, we need another definition and lemma.

Definition 6. *For every criticality situation $\theta \in \Theta$ and all $S \subseteq N$, function $p(\gamma_S^\theta)$ is defined as*

$$p(\gamma_S^\theta) = \frac{\mu^s \lambda^{n-s}}{(\lambda + \mu)^n}.$$

Lemma 7. *For any pair of criticality situations $\theta, \theta' \in \Theta$ with $N \subset N'$, $\lambda = \lambda'$, $\mu = \mu'$, $d_i = d_i'$ for all $i \in N$ and $h = h'$, it holds that*

$$\begin{aligned} v^\theta(M) &= v^{\theta'}(M) && \text{for all } M \subseteq N, \\ v^{\gamma_S^\theta}(M) &= v^{\gamma_S^{\theta'}}(M) && \text{for all } M \subseteq N \text{ and all } S \subseteq N. \end{aligned}$$

Note that the result of Lemma 7 follows directly, as the value of any coalition $M \subseteq N$ depends on the parameters of players $i \in M$ only.

Theorem 3. *For every criticality situation $\theta \in \Theta$ and corresponding criticality game (N, v^θ), it holds that*

$$v^\theta = \sum_{S:S\subseteq N} p(\gamma_S^\theta) \cdot v^{\gamma_S^\theta}. \tag{6}$$

Proof : See Appendix.

We illustrate the result of Theorem 3 in the following example.

Example 7. *Consider the situation of Example 6. Then*

$$p(\gamma_S^\theta) = \begin{cases} \frac{8}{27} & \text{if} \quad |S| = 3 \\ \frac{4}{27} & \text{if} \quad |S| = 2 \\ \frac{2}{27} & \text{if} \quad |S| = 1 \\ \frac{1}{27} & \text{if} \quad |S| = 0. \end{cases}$$

For example, for coalition $M = \{2,3\}$, we obtain, using Table 4 and Theorem 3

$$v^\theta(\{2,3\}) = \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot (0 + 0 + (2-1)) + \frac{4}{27} \cdot (0 + (2-1) + 0) + \frac{8}{27} \cdot 0 = \frac{2}{9}.$$

It is easily checked that all other values can be determined via Table 4 as well. ◇

The following Lemma formally expresses that a game, which is a convex combination of several games with a non–empty core, still has a non–empty core.

Lemma 8. *Let $(v^k)_{k=1}^l$ be a set of games with the same player set N, and $x^k \in C(N, v_k)$ for $k = 1, 2, \ldots, l$. Then, it holds for all $(\alpha_k)_{k=1}^l \in \mathbb{R}_+^l$ that*

$$y = \sum_{k=1}^{l} \alpha_k \cdot x^k \in C(N, z),$$

where $z = \sum_{k=1}^l \alpha_k \cdot v^k$.

Using Lemma 8, we conclude that every criticality game has a non–empty core.

Theorem 4. *Every criticality game has a non–empty core.*

Proof : Let $\theta \in \Theta$ be a criticality situation and (N, v^θ) be the corresponding criticality game. Moreover, for every BBHM situation γ_S^θ with $S \subseteq N$ it holds, based on Theorem 1, that the associated BBHM game $(N, v^{\gamma_S^\theta})$ has a non–empty core. Then, by Lemma 8 and Theorem 3, the associated criticality game (N, v^θ) has a non–empty core. □

A next and natural step is to investigate wether criticality games are totally balanced, i.e., every sub criticality game has a non–empty core.

Theorem 5. *Every criticality game is totally balanced.*

Proof : Let $\theta \in \Theta$ be a criticality situation and (N, v^θ) be the corresponding criticality game. Let $M \subseteq N$ and consider subgame (M, v_M^θ) with $v_M^\theta(K) = v^\theta(K)$ for all $K \subseteq M$. Using Lemma 7 there exists a $\theta' \in \Theta$ with $N' = M$, $\lambda' = \lambda$, $\mu' = \mu$, and $d_i' = d_i$ for all $i \in N'$ for which holds that $v^\theta(K) = v^{\theta'}(K)$ for all $K \subseteq N'$. Hence, $v_M^\theta(K) = v^{\theta'}(K)$ for all $K \subseteq M$. Using Theorem 4 $(N', v^{\theta'})$ is balanced and so (M, v_M^θ) is balanced as well. Criticality games are totally balanced. □

It is well–known (see e.g. Peleg and Sudhölter [14]) that totally balanced games are superadditive. Hence, criticality games are superadditive, i.e., the costs savings of any two disjoint coalitions are less than or equal to the cost savings of the union of those two disjoint coalitions. In addition, as criticality games are zero–normalized, the cost savings of any coalition are at least the cost savings of any of its subcoalitions, i.e., criticality games are monotonic.

Corollary 2. *Every criticality game is superadditive and monotonic.*

4.3 A class of allocation rules

An allocation rule on criticality situations is defined as a mapping f that assigns to any criticality situation $\theta = (N, \mu, \lambda, d, h) \in \Theta$ a vector $f(\theta) \in \mathbb{R}^N$. In section 4.2 we learned that criticality games can be recognized as convex combinations of BBHM games. In what follows, we will construct a class of allocation rules that exploits this. Recall from Section 2 that any core-element of a BBHM situation $\gamma \in \Gamma$ can be described as $\mathscr{L}(\gamma, \alpha)$ for some $\alpha \in [0, 1]$. Now, for any $\theta \in \Theta$ and all related BBHM situations, i.e., all possible sets of sellers, we set $\alpha \in [0, 1]$, i.e., we split the (trading) profit between the active buyers and active sellers in such a way that it is a core member of the corresponding BBHM game. Then, we multiply these outcomes with the probability that these BBHM situations occur and finally we add these terms. As for every possible set of sellers $S \subseteq N$ we have to select a parameter, we can formulate a whole class of allocation rules.

Definition 7. *For all $\widehat{\alpha} \in [0, 1]^{2^N}$ allocation rule $\mathscr{B}^{\widehat{\alpha}}$ assigns to any criticality situation $\theta = (N, \mu, \lambda, d, h) \in \Theta$ allocation*

$$\mathscr{B}^{\widehat{\alpha}}(\theta) = \sum_{S \subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \widehat{\alpha}_S). \tag{7}$$

We illustrate such an allocation rule in Example 8.

Example 8. *Consider the situation of Example 2. In Table 5, the extreme prices $\underline{\delta}$ and $\overline{\delta}$ are presented for any γ_S^θ with $S \subseteq N$. Note that the extreme prices for $S = \{2\}$ are presented in Example 1 as well.*

Table 5: Corresponding extreme buyer (seller) prices

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\overline{\delta}$	0	0	5	5	0	2	2	0
$\underline{\delta}$	0	0	2	2	0	1	1	0

Let $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. This fixes a core-element for all $(N, v^{\gamma_S^\theta})$ with $S \subseteq N$. In Table 6, all these core-elements are presented.

Table 6: Corresponding core elements for BBHM situations

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mathscr{L}_1(\gamma_S^\theta, \frac{1}{2})$	0	0	$1\frac{1}{2}$	$1\frac{1}{2}$	0	0	$3\frac{1}{2}$	0
$\mathscr{L}_2(\gamma_S^\theta, \frac{1}{2})$	0	0	$1\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
$\mathscr{L}_3(\gamma_S^\theta, \frac{1}{2})$	0	0	0	$2\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0

For example, given that $S = \{2\}$, the payoff of player 1 is

$$\mathscr{L}_1\left(\gamma_{\{2\}}^\theta, \frac{1}{2}\right) = \max\{5 - (2 + \frac{1}{2} \cdot (5-2)), 0\} = 1\frac{1}{2}.$$

Now, by combining terms, we obtain

$$\begin{aligned}
\mathscr{B}_1^{\widehat{\alpha}}(\theta) &= \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot \left(1\frac{1}{2} + 1\frac{1}{2}\right) + \frac{4}{27} \cdot 3\frac{1}{2} + \frac{8}{27} \cdot 0 = \frac{20}{27} \\
\mathscr{B}_2^{\widehat{\alpha}}(\theta) &= \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot 1\frac{1}{2} + \frac{4}{27} \cdot \frac{1}{2} + \frac{8}{27} \cdot 0 = \frac{5}{27} \\
\mathscr{B}_3^{\widehat{\alpha}}(\theta) &= \frac{1}{27} \cdot 0 + \frac{2}{27} \cdot 2\frac{1}{2} + \frac{4}{27} \cdot \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{8}{27} \cdot 0 = \frac{9}{27}.
\end{aligned}$$

It is easily checked that $(\frac{20}{37}, \frac{5}{27}, \frac{9}{27}) \in C(N, v^\theta)$. ◇

The next result follows directly from Theorem 1, Lemma 8, and Theorem 3.

Corollary 3. *For every criticality situation $\theta \in \Theta$ it holds for any $\widehat{\alpha} \in [0,1]^{2^N}$ that $\mathscr{B}^{\widehat{\alpha}}(\theta) \in C(N, v^\theta)$.*

One may wonder whether every element of the core of a criticality game results from an allocation rule $\mathscr{B}^{\widehat{\alpha}}$ for some $\widehat{\alpha} \in [0,1]^{2^N}$, i.e., if $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} = C(N, v^\theta)$. The following example shows that this is not the case in general.

Example 9. *Consider the situation of Example 2 and $x = (1\frac{1}{27}, \frac{6}{27}, 0) \in C(N, v^\theta)$. Let $\widehat{\alpha} \in [0,1]^{2^N}$, then*

$$\mathscr{L}_3\left(\gamma_{\{3\}}^\theta, \widehat{\alpha}_{\{3\}}\right) = \underline{\delta} + \widehat{\alpha}_{\{3\}}(\overline{\delta} - \underline{\delta}) - \lambda \cdot d_3 = 1 + 3 \cdot \widehat{\alpha}_{\{3\}} > 0,$$

as $\widehat{\alpha}_{\{3\}} \in [0,1]$. In addition, $p\left(\gamma_{\{3\}}^\theta\right) = \frac{2}{27} > 0$ and thus $\mathscr{B}_3^{\widehat{\alpha}}(\theta) > 0$.

We conclude that $x \neq (\mathscr{B}_i^{\widehat{\alpha}}(\theta))_{i \in N}$ for all $\widehat{\alpha} \in [0,1]^{2^N}$. A graphical representation of $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ and the core is represented in Figure 1. ◇

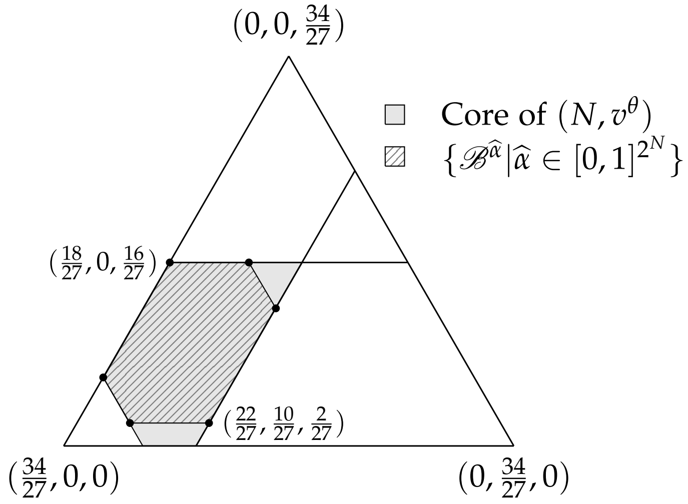

Figure 1: Core and $\{\mathscr{B}^{\widehat{\alpha}}|\widehat{\alpha}\in[0,1]^{2^N}\}$ of game (N,v^θ)

Figure 1 shows that $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha}\in[0,1]^{2^N} \right\}$ is a convex set[2] and is spanned by six vectors only. These vectors turn out to be special allocations. For example, vector $(\frac{18}{27},0,\frac{16}{27})$ is the allocation where, based on order (3,1,2), a player selects core allocations in its own best interest of BBHM games in which he is active under the restriction that the players before him in the corresponding order recursively selected their related core allocations. For example, for order (3,1,2), first player 3 will select core allocations of BBHM games $(N,v^{\gamma^\theta_{\{3\}}})$, $(N,v^{\gamma^\theta_{\{1,3\}}})$ and $(N,v^{\gamma^\theta_{\{2,3\}}})$. As player 3 is an active seller in all of these BBHM games, player 3 will select three times the *seller optimal* core allocation. Subsequently player 1 has the right to select the core allocation of the remaining BBHM game $(N,v^{\gamma^\theta_{\{2\}}})$. As player 1 is an active buyer in this BBHM game, player 1 will select the *buyer optimal* core allocation. In a similar way, the other vectors can be obtained by considering the different orders of the player set. Note that for BBHM games without active players, and so no trading profit, there is no need to select a core allocation.

Now, we will formalize these vectors. An order on N is a bijection $\omega:\{1,2,\ldots,|N|\}\to N$, where $\omega(j)$ indicates which player is in position j and $\omega^{-1}(i)$ indicates the position of player i. The set of all orders is denoted by Ω^N. For any $S\subseteq N$ let l_S be the number of active sellers (buyers) that trade a good in

[2]One can check easily that this set is convex by the linearity of $\mathscr{B}^{\widehat{\alpha}}$ in $\widehat{\alpha}$. See also the first part of the proof of Theorem 7.

γ^θ_S. For any $S \subseteq N$ with at least one active seller in γ^θ_S, i.e., $l_S \geq 1$, let $N^S = \left\{\sigma_B^{-1}(i) \;\mid\; 1 \leq i \leq l_S\right\} \cup \left\{\sigma_S^{-1}(i) \;\mid\; s - l_S + 1 \leq i \leq s\right\}$ be the subset of N with active sellers and buyers, and let $i^*_S \in N^S$ such that $\omega^{-1}(i^*_S) \leq \omega^{-1}(i)$ for all $i \in N^S$, i.e., i^*_S the first active player according to ω. Then, for any $S \subseteq N$ with $l_S \geq 1$ we define

$$\alpha^\omega_S = \begin{cases} 1 & \text{if } i^*_S \in S \\ \\ 0 & \text{if } i^*_S \in B. \end{cases}$$

Hence, if the first active player is a seller, it selects the *seller optimal* core allocation and if the first active player is a buyer, it selects the *buyer optimal core* allocation. For any $S \subseteq N$ for which $l_S = 0$, and thus $\mathscr{L}_i(\gamma^\theta_S, \alpha_S) = 0$ for all $i \in N$ and any $\alpha_S \in [0,1]$, we set (arbitrarily) $\alpha^\omega_S = \frac{1}{2}$. For any $\omega \in \Omega$ this fixes vector $\alpha^\omega = (\alpha^\omega_S)_{S \subseteq N}$. We call these vectors *extreme BBHM payoff vectors*.

Example 10. *Consider the situation of Example 2. For order* $(2,1,3)$*, the first active player in BBHM game* $(N, v^{\gamma^\theta_{\{2\}}})$ *is a seller, namely player 2, and as a consequence* $\alpha^{(2,1,3)}_{\{2\}} = 1$*. In Table 5 all extreme BBHM payoff vectors are presented.* ◇

Table 7: All extreme BBHM payoff vectors

ω	$\alpha^\omega_\emptyset$	$\alpha^\omega_{\{1\}}$	$\alpha^\omega_{\{2\}}$	$\alpha^\omega_{\{3\}}$	$\alpha^\omega_{\{1,2\}}$	$\alpha^\omega_{\{1,3\}}$	$\alpha^\omega_{\{2,3\}}$	$\alpha^\omega_{\{1,2,3\}}$
(1,2,3)	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(1,3,2)	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
(2,1,3)	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(2,3,1)	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(3,1,2)	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$
(3,2,1)	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$

In line with the Shapley value (Shapley [18]) as well as the Alexia value (Tijs et al. [22]), it is a natural choice to consider the average over all extreme BBHM payoff vectors.

Example 11. *Consider the situation of Example 2. In Table 8* $\mathscr{B}^{\alpha^\omega}$ *is represented for all* $\omega \in \Omega$ *as well as the average of these vectors. In addition, the Shapley value is given by* $\left(\frac{49}{81}, \frac{22}{81}, \frac{31}{81}\right)$ *and the Alexia value by* $\left(\frac{19}{27}, \frac{6}{27}, \frac{9}{27}\right)$*.* ◇

Table 8: $\mathscr{B}^{\alpha^{\omega}}$ for all $\omega \in \Omega$ and average

ω	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)	average
$\mathscr{B}_1^{\widehat{\alpha}^{\omega}}(\theta)$	$\frac{28}{27}$	$\frac{28}{27}$	$\frac{22}{27}$	$\frac{12}{27}$	$\frac{18}{27}$	$\frac{12}{27}$	$\frac{20}{27}$
$\mathscr{B}_2^{\widehat{\alpha}^{\omega}}(\theta)$	$\frac{4}{27}$	0	$\frac{10}{27}$	$\frac{10}{27}$	0	$\frac{6}{27}$	$\frac{5}{27}$
$\mathscr{B}_3^{\widehat{\alpha}^{\omega}}(\theta)$	$\frac{2}{27}$	$\frac{6}{27}$	$\frac{2}{27}$	$\frac{12}{27}$	$\frac{16}{27}$	$\frac{16}{27}$	$\frac{9}{27}$

Note that the average of $\mathscr{B}^{\alpha^{\omega}}$ deviates from the Shapley value and the Alexia value. However, it coincides with the payoff vector of (the simple) allocation rule $\mathscr{B}^{\widehat{\alpha}}(\theta)$ with $\alpha_S = \frac{1}{2}$ for all $S \subseteq N$. This is no coincidence.

Theorem 6. *Let $\theta \in \Theta$ and $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. Then*

$$\mathscr{B}^{\widehat{\alpha}}(\theta) = \frac{1}{n!} \sum_{\omega \in \Omega} \mathscr{B}^{\alpha^{\omega}}(\theta).$$

Proof : See Appendix.

Recall that Figure 2 showed us that $\{\mathscr{B}^{\widehat{\alpha}} | \widehat{\alpha} \in [0,1]^{2^N}\}$ is spanned by the extreme BBHM payoff vectors. This does not holds in general.

Theorem 7. *For any $\theta \in \Theta$ it holds that*

$$\textit{convexhull}\left\{ \mathscr{B}^{\widehat{\alpha}^{\omega}} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{B}^{\widehat{\alpha}} | \widehat{\alpha} \in [0,1]^{2^N} \right\}$$

and there exists a $\theta \in \Theta$ for which this relation is strict.
Proof : See Appendix.

We conclude by showing that the average of the payoff vectors resulting from all allocation rules with $\alpha \in \{0,1\}^{2^N}$ coincides with the average of the extreme BBHM payoff vectors.

Theorem 8. *Let $\theta \in \Theta$ and $\widehat{\alpha}_S = \frac{1}{2}$ for all $S \subseteq N$. Then*

$$\mathscr{B}^{\widehat{\alpha}}(\theta) = \frac{1}{2^{2^N}} \sum_{\tau \in \{0,1\}^{2^N}} \mathscr{B}^{\tau}(\theta).$$

Proof : See Appendix.

5 Conclusions

In this article, an environment is considered in which several independent service providers with possible criticality differences can cooperate by pooling their (single) spare part according to a one–by–one critical level policy. An associated cooperative game, called criticality game, is formulated to investigate the allocation of the collective costs savings. We presented necessary and sufficient conditions to ensure convexity of criticality games. Thereafter, we showed that values of criticality games may be recognized as convex combinations of values of Böhm–Bahwerk Horse Market games and as a consequence we could prove that criticality games are (totally) balanced. In addition, we presented a class of allocation rules for which the payoff vectors are core members. Finally, we showed that within this class of allocation rules, there exists a quite intuitive and simple allocation rule for which the related payoff vector coincides with the average of the extreme BBHM payoff vectors. The main managerial insight of this paper is that cooperation is beneficial in our spare parts setting and can be supported by a stable allocation. Although our focus is spare parts oriented, our results may be applicable to other resource systems (where each player owns a single resource and the intensity of the demand is similar for all players) as well. Our current work can be extended to situations with (i) players having an arbitrary number of spare parts on stock (rather than one), (ii) arrival intensities that may differ per player, (iii) non-zero lateral transshipment costs or (iv) an even smarter pooling approach, i.e., a critical level policy (for every coalition) that minimizes the long-term average costs, rather than the fixed one-by-one critical level studied in this paper.

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Appendix

<u>Proof Lemma 2</u>

Let $\theta \in \Theta$ be a criticality situation and $M \subseteq N$. Then the sum of the holding costs of all individual players of coalition M is given by

$$\begin{aligned}
h \cdot m \cdot \pi(1,1) &= h \cdot m \cdot \frac{\mu}{\mu+\lambda} \\
&= h \cdot m \cdot \frac{\mu}{(\mu+\lambda)^m} \cdot (\mu+\lambda)^{m-1} \\
&= h \cdot m \cdot \frac{\mu}{(\mu+\lambda)^m} \cdot \sum_{i=0}^{m-1} \binom{m-1}{i} \mu^i \lambda^{m-i-1} \\
&= h \cdot m \cdot \frac{1}{(\mu+\lambda)^m} \cdot \sum_{i=0}^{m-1} \binom{m-1}{i} \mu^{i+1} \lambda^{m-(i+1)} \\
&= h \cdot m \cdot \frac{1}{(\mu+\lambda)^m} \cdot \sum_{i=1}^{m} \binom{m-1}{i-1} \mu^{i} \lambda^{m-i} \\
&= h \cdot \frac{1}{(\mu+\lambda)^m} \cdot \sum_{i=1}^{m} \frac{m(m-1)!}{(i-1)!(m-i)!} \mu^{i} \lambda^{m-i} \\
&= h \cdot \frac{1}{(\mu+\lambda)^m} \cdot \sum_{i=1}^{m} \frac{m(m-1)!}{(i-1)!(m-i)!} \frac{i}{i} \mu^{i} \lambda^{m-i} \\
&= h \cdot \frac{1}{(\mu+\lambda)^m} \cdot \sum_{i=1}^{m} \binom{m}{i} \mu^{i} \lambda^{m-i} \cdot i \\
&= h \cdot \sum_{i=1}^{m} \binom{m}{i} \left(\frac{\mu}{\lambda+\mu}\right)^{i} \left(\frac{\lambda}{\lambda+\mu}\right)^{m-i} \cdot i \\
&= h \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i.
\end{aligned}$$

The first and last equality hold by definition. The third equality holds by the binomium of Newton. The other equalities hold by rewriting. □

<u>Proof of Lemma 3</u>

Let $\theta \in \Theta$ be a criticality situation and $M \subseteq N$. First, observe that

$$
\begin{aligned}
\pi(1,0) &= 1 - \pi(1,1) \\
&= 1 - \frac{1}{m} \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i \\
&= \frac{1}{m} \left[m - \sum_{i=1}^{m} \pi(m,i) \cdot i \right] \\
&= \frac{1}{m} \left[\sum_{i=0}^{m} \pi(m,i) \cdot m - \sum_{i=1}^{m} \pi(m,i) \cdot i \right] \\
&= \frac{1}{m} \left[\sum_{i=0}^{m} \pi(m,i) \cdot m - \sum_{i=0}^{m} \pi(m,i) \cdot i \right] \\
&= \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m} \cdot (m-i) \right],
\end{aligned} \tag{8}
$$

where the first equality holds as $\pi(1,0) + \pi(1,1) = 1$. The second equality holds by applying Lemma 2 with $h = 1$. The third equality holds by rewriting terms. The fourth equality holds as $\sum_{i=0}^{m} \pi(m,i) = 1$. The fifth equality holds by adding zero, i.e., by adding $\pi(m,0) \cdot 0 = 0$. The last equality holds by combining summations and some rewriting.

Secondly, observe that

$$
\begin{aligned}
\sum_{i \in M} c^{\theta}(\{i\}) &= \sum_{i \in M} \left[\pi(1,0) \cdot \lambda \cdot d_{\sigma^{-1}_{\{i\}}(1)} + \pi(1,1) \cdot h \right] \\
&= \sum_{i \in M} \left[\pi(1,0) \cdot \lambda \cdot d_{\sigma^{-1}_{\{i\}}(1)} \right] + m \cdot \pi(1,1) \cdot h \\
&= \lambda \cdot \pi(1,0) \cdot \sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + m \cdot \pi(1,1) \cdot h \\
&= \lambda \cdot \pi(1,0) \cdot \sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + h \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i \\
&= \lambda \cdot \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m} \cdot (m-i) \right] \cdot \sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)} + h \cdot \sum_{i=1}^{m} \pi(m,i) \cdot i,
\end{aligned}
$$

where the first equality holds by definition. The second equality holds as $\pi(1,1) \cdot h$ is independent of i and therefore the summation can be replaced by a factor m upfront. The third equality holds, as $\pi(1,0) \cdot \lambda$ is independent of i and $\sum_{i \in M} d_{\sigma^{-1}_{\{i\}}(1)} = \sum_{j=1}^{m} d_{\sigma^{-1}_{M}(j)}$. The fourth equality holds by applying Lemma 2. The last equality holds by the last expression of (8).

Finally, observe that

$$
\begin{aligned}
&\sum_{i\in M} c^{\theta}(\{i\}) - c^{\theta}(M)\\
=&\lambda \cdot \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m}\cdot(m-i)\right]\cdot \sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} + h\cdot \sum_{i=1}^{m} \pi(m,i)\cdot i\\
&-\left(\sum_{i=0}^{m}\left[\pi(m,i)\cdot\left(i\cdot h+\lambda\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\right)\\
=&\lambda \cdot \left[\sum_{i=0}^{m} \frac{\pi(m,i)}{m}\cdot(m-i)\right]\cdot \sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} - \lambda\cdot \sum_{i=0}^{m}\left[\pi(m,i)\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right]\\
=&\lambda \cdot \sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} - \frac{m}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\\
=&\lambda \cdot \sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} - \frac{m+i-i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\\
=&\lambda \cdot \sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{m} d_{\sigma_M^{-1}(j)} - \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\\
=&\lambda \cdot \sum_{i=0}^{m}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\\
=&\lambda \cdot \sum_{i=1}^{m-1}\left[\frac{\pi(m,i)}{m}\cdot(m-i)\cdot\left(\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right]\\
=&\lambda \cdot \sum_{i=1}^{m-1}\left[\pi(m,i)\cdot\left(\frac{m-i}{m}\cdot\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)}\right)\right],
\end{aligned}
$$

where the first equality holds by definition. The second equality holds as $h \cdot \sum_{i=1}^{m} \pi(m,i)\cdot i - (\sum_{i=0}^{m}\pi(m,i)\cdot i\cdot h) = 0$. In the third equality, we write $\lambda \cdot \left[\sum_{i=0}^{m}\frac{\pi(m,i)}{m}\cdot(m-i)\right]$ in front of all terms. In the fourth equality we add zero. In the fifth equality we split the fraction in front of the last summation into two parts. The sixth equality holds by combining the second and third summation. The seventh equality holds as for $i=0$, $\sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m-i}\cdot\sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} = 0$ and for $i=m$, $(m-i)=0$. The last equality holds by rewriting. □

Proof Lemma 4

Let Θ be a criticality situation and $M \subseteq N$. By Lemma 3, the value of coalition M of the corresponding criticality game is given by

$$v^{\theta}(M) = \lambda \cdot \sum_{i=1}^{m-1} \left[\pi(m,i) \cdot \left(\frac{m-i}{m} \cdot \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \cdot \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} \right) \right].$$

As $\lambda, \mu > 0$, it is sufficient to check whether the last part of the summation is larger than or equal to zero. Now, take $i \in \{1, 2, ..., m-1\}$, then it holds

$$\begin{aligned}
\frac{m-i}{m} \sum_{j=1}^{i} d_{\sigma_M^{-1}(j)} - \frac{i}{m} \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(j)} &\geq \frac{m-i}{m} \sum_{j=1}^{i} d_{\sigma_M^{-1}(i)} - \frac{i}{m} \sum_{j=i+1}^{m} d_{\sigma_M^{-1}(i+1)} \\
&= \frac{m-i}{m} \cdot i \cdot d_{\sigma_M^{-1}(i)} - \frac{i}{m} \cdot (m-i) \cdot d_{\sigma_M^{-1}(i+1)} \\
&= \frac{m-i}{m} \cdot i \cdot \left(d_{\sigma_M^{-1}(i)} - d_{\sigma_M^{-1}(i+1)} \right) \\
&\geq 0.
\end{aligned}$$

where the first inequality holds as $d_{\sigma_M^{-1}(j)} \geq d_{\sigma_M^{-1}(i)}$ for all $j \in N$ with $j \leq i$ and $d_{\sigma_M^{-1}(j)} \leq d_{\sigma_M^{-1}(i+1)}$ for all $j \in N$ with $j \geq i+1$. The first equality holds as $d_{\sigma_M^{-1}(i)}$ and $d_{\sigma_M^{-1}(i+1)}$ are independent of j and thus the summations can be replaced by a factor i and $(m-i)$ upfront, respectively. The second equality holds by some rewriting. The last inequality holds as $((m-i)/m) \cdot i \geq 0$ and $d_{\sigma_M^{-1}(i)} \geq d_{\sigma_M^{-1}(i+1)}$ for all $i \in \{1, 2, \ldots, m-1\}$ and any $M \subseteq N$. This concludes the proof. □

Proof of Theorem 2

We distinguish $|N| = 3$ and $|N| \geq 4$. Without loss on generality, let $\theta \in \Theta$ be a criticality situation with $N = \{1, 2, 3\}$[3]. (⇒) Suppose that the game is convex. We will show that $d_1 - 2d_2 + d_3 = 0$. By convexity, it holds that

$$v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,3\}) - \left(v^{\theta}(\{1,2\}) - v^{\theta}(\{1\}) \right) \geq 0.$$

By using Lemma 3, we obtain

$$\begin{aligned}
&v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,3\}) - \left(v^{\theta}(\{12\}) - v^{\theta}(\{1\}) \right) \\
=&\left(\frac{\lambda}{(\lambda+\mu)^3} \left[\lambda^2\mu(2d_1 - d_2 - d_3) + \lambda\mu^2(d_1 + d_2 - 2d_3) - (\lambda\mu(\lambda+\mu))(d_1 - d_3) \right] \right) \\
&- \left(\frac{\lambda}{(\lambda+\mu)^3} \left[(\lambda\mu(\lambda+\mu))(d_1 - d_2) \right] - 0 \right) \\
=&\lambda \frac{\lambda\mu^2}{(\lambda+\mu)^3} (-d_1 + 2d_2 - d_3).
\end{aligned}$$

[3]Note that the same reasoning holds for any N with $|N| = 3$.

As $\lambda, \mu > 0$ it should (thus) hold that

$$d_1 - 2d_2 + d_3 \leq 0. \tag{9}$$

Moreover, it holds that

$$v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) \geq 0$$

By using Lemma 3, we obtain

$$\begin{aligned}
&v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) \\
=&\left(\frac{\lambda}{(\lambda+\mu)^3}\Big[\lambda^2\mu(2d_1 - d_2 - d_3) + \lambda\mu^2(d_1 + d_2 - 2d_3) - (\lambda\mu(\lambda+\mu)(d_2 - d_3)\Big]\right) \\
&-\left(\frac{\lambda}{(\lambda+\mu)^3}\left[(\lambda\mu(\lambda+\mu))(d_1 - d_3)\right] - 0\right) \\
=&\lambda\frac{\lambda^2\mu}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3).
\end{aligned}$$

As $\lambda, \mu > 0$ it should (thus) hold that

$$d_1 - 2d_2 + d_3 \geq 0. \tag{10}$$

As (9) and (10) should hold both, we conclude that

$$d_1 - 2d_2 + d_3 = 0.$$

($\Leftarrow$) Suppose that $d_1 - 2d_2 + d_3 = 0$. We will show that $v^\theta(T \cup \{i\}) - v^\theta(T) - (v^\theta(S \cup \{i\}) - v^\theta(S)) \geq 0$ for all $S, T \subseteq \{1,2,3\} \backslash \{i\}$ and all $i \in \{1,2,3\}$. From the only–if–part we know that

$$v^\theta(\{1,2,3\}) - v^\theta(\{1,3\}) - \Big(v^\theta(\{1,2\}) - v^\theta(\{1\})\Big) = 0,$$

$$v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,3\}) - v^\theta(\{3\})\Big) = 0.$$

By using Lemma 3, we obtain

$$\begin{aligned}
&v^\theta(\{1,2,3\}) - v^\theta(\{2,3\}) - \Big(v^\theta(\{1,2\}) - v^\theta(\{2\})\Big) \\
&= \lambda\frac{\lambda\mu}{(\lambda+\mu)^3} \cdot ((d_1 - d_2)\lambda + (d_2 - d_3)\mu) \geq 0,
\end{aligned}$$

where the inequality holds as $\lambda, \mu > 0$ and $d_1 \geq d_2 \geq d_3$. By Lemma 3, we obtain

$$v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,2\}) - \Big(v^{\theta}(\{2,3\}) - v^{\theta}(\{2\})\Big)$$
$$= \lambda \frac{\lambda\mu}{(\lambda+\mu)^3} \cdot ((d_1 - d_2)\lambda + (d_2 - d_3)\mu) \geq 0,$$

where the inequality holds as $\lambda, \mu > 0$ and $d_1 \geq d_2 \geq d_3$. By Lemma 3, we also obtain

$$v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,2\}) - \Big(v^{\theta}(\{1,3\}) - v^{\theta}(\{1\})\Big)$$
$$= -\lambda \frac{\lambda\mu^2}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3) = 0,$$

where the equality holds as $d_1 - 2d_2 + d_3 = 0$. By Lemma 3, we once again obtain

$$v^{\theta}(\{1,2,3\}) - v^{\theta}(\{1,3\}) - \Big(v^{\theta}(\{2,3\}) - v^{\theta}(\{3\})\Big)$$
$$= \lambda \frac{\lambda^2\mu}{(\lambda+\mu)^3}(d_1 - 2d_2 + d_3) = 0,$$

where the inequality holds as $d_1 - 2d_2 + d_3 = 0$. Finally, let $i, j \in N$ with $i \neq j$. Then

$$v^{\theta}(\{i,j\}) - v^{\theta}(\{i\}) - (v^{\theta}(\{j\}) - v^{\theta}(\emptyset) = v\{i,j\}) \geq 0,$$

where the equality holds as $v^{\theta}(\{i\}) = v^{\theta}(\{j\}) = v^{\theta}(\emptyset) = 0$. The inequality holds by Lemma 4. Hence, $v^{\theta}(T \cup \{i\}) - v^{\theta}(T) - (v^{\theta}(S \cup \{i\}) - v^{\theta}(S)) \geq 0$ for all $S, T \subseteq \{1,2,3\} \backslash \{i\}$ and all $i \in \{1,2,3\}$ and thus we can conclude that the game is convex.

Now, let $\theta \in \Theta$ be a criticality situation with $|N| \geq 4$.

($\Rightarrow$) Suppose that the game is convex. As a consequence, any subgame with player set N' where $N' \subset N$ and $|N'| = 3$ is convex as well. From this, we conclude by the first part of this proof that

$$d_i - 2d_j + d_k = 0 \quad \forall i, j, k \in N \text{ with } i < j < k.$$

($\Leftarrow$) Suppose that $d_i - 2d_j - d_k = 0$ for all $i, j, k \in N$ with $i < j < k$. From $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(2)} - d_{\sigma_N^{-1}(k-1)} = 0$ and $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(k)} = 0$ with $k \geq 4$ ($k \in N$), we can immediately conclude that $d_{\sigma_N^{-1}(3)} = d_{\sigma_N^{-1}(4)} = \ldots = d_{\sigma_N^{-1}(n)}$. From $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ and $d_{\sigma_N^{-1}(2)} - 2 \cdot d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ we can conclude that $d_{\sigma_N^{-1}(1)} = d_{\sigma_N^{-1}(2)}$ and from $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(2)} + d_{\sigma_N^{-1}(4)} = 0$ and $d_{\sigma_N^{-1}(1)} - 2 \cdot d_{\sigma_N^{-1}(3)} + d_{\sigma_N^{-1}(4)} = 0$ we can conclude that $d_{\sigma_N^{-1}(2)} = d_{\sigma_N^{-1}(3)}$. Hence, for every possible solution, it should hold that $d_{\sigma_N^{-1}(1)} = d_{\sigma_N^{-1}(2)} = \ldots = d_{\sigma_N^{-1}(n)}$. Now, we will show that $(d_i)_{i \in N}$ with $d_i = z \in \mathbb{R}_+$ for all $i \in N$ is a feasible solution. Let $i, j, k \in N$ with $i < j < k$. Then, $d_{\sigma_N^{-1}(i)} - 2 \cdot d_{\sigma_N^{-1}(j)} + d_{\sigma_N^{-1}(k)} = z - 2z + z = 0$. Hence, $d_i = z \in \mathbb{R}_+$ is indeed feasible. For this feasible solution, it holds that all downtime costs are similar and thus no cost savings are obtained in the corresponding game, i.e., $v^\theta(M) = 0$ for all $M \subseteq N$. We conclude that the game is additive and as a consequence the game is convex. □

Proof of Lemma 6

Let $\theta \in \Theta$ be a criticality situation and $k \in \mathbb{N}$ such that $1 \leq k \leq n-1$. Based on Lemma 5 it holds that player $j \in N$ is an active buyer in criticality situation γ_S^θ with $|S| = k$ if and only if $1 \leq \sigma_N^{-1}(j) \leq k$. Let player $j \in N$ be a buyer with $1 \leq \sigma_N^{-1}(j) \leq k$. It is easily seen that, from the $\binom{n}{k}$ BBHM games with k sellers, $\binom{n-1}{k}$ times player j is an active buyer. This results in a total value of $\binom{n-1}{k} \cdot \lambda \cdot d_{\sigma_N^{-1}(j)}$. Similarly, player $j \in N$ is an active seller in criticality situation γ_S^θ with $|S| = k$ if and only if $k+1 \leq \sigma_N^{-1}(j) \leq n$. Let player $j \in N$ be a seller and $k+1 \leq \sigma_N^{-1}(j) \leq n$. It is easily seen that, from the $\binom{n}{k}$ BBHM games, $\binom{n-1}{k-1}$ times player j is an active seller. This results in a total costs of $\binom{n-1}{k-1} \cdot \lambda \cdot d_{\sigma_N^{-1}(j)}$. The total joint profit of all BBHM games with k sellers is

$$
\begin{aligned}
\sum_{S \subseteq N: |S| = k} v^{\gamma_S^\theta}(N) &= \binom{n-1}{k} \sum_{j=1}^{k} \lambda \cdot d_{\sigma_N^{-1}(j)} - \binom{n-1}{k-1} \sum_{j=k+1}^{n} \lambda \cdot d_{\sigma_N^{-1}(j)}. \\
&= \lambda \cdot \frac{(n-1)!}{(k-1)!(n-1-k)!} \left(\frac{1}{k} \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=k+1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \cdot \binom{n-1}{k-1} \left(\left(\frac{1}{k} + \frac{1}{n-k} \right) \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \cdot \binom{n-1}{k-1} \left(\frac{n}{k(n-k)} \sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{1}{n-k} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right) \\
&= \lambda \cdot \binom{n}{k} \left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)} - \frac{k}{n} \sum_{j=1}^{n} d_{\sigma_N^{-1}(j)} \right),
\end{aligned}
$$

where the first equality follows from the explanation above. The second equality holds by writing λ and part of the binomial coefficient upfront. The third equality holds by adding zero. The last equality holds by some rewriting. $\square$

Proof of Theorem 3

Let $\theta \in \Theta$ be a criticality situation. Then

$$\begin{aligned}
&\sum_{S\subseteq N} p(\gamma_S^\theta)\cdot v^{\gamma_S^\theta}(N)\\
=&\sum_{k=1}^{n-1}\left[\sum_{S\subseteq N:|S|=k}\left(\frac{\mu}{\lambda+\mu}\right)^k\cdot\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k}\cdot v^{\gamma_S^\theta}(N)\right]\\
=&\sum_{k=1}^{n-1}\left[\left(\frac{\mu}{\lambda+\mu}\right)^k\cdot\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k}\cdot\sum_{S\subseteq N:|S|=k} v^{\gamma_S^\theta}(N)\right]\\
=&\sum_{k=1}^{n-1}\left[\left(\frac{\mu}{\lambda+\mu}\right)^k\cdot\left(\frac{\lambda}{\lambda+\mu}\right)^{n-k}\cdot\lambda\cdot\binom{n}{k}\cdot\left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)}-\frac{k}{n}\sum_{j=1}^{n} d_{\sigma_N^{-1}(j)}\right)\right]\\
=&\lambda\cdot\sum_{k=1}^{n-1}\left[\pi(n,k)\cdot\left(\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)}-\frac{k}{n}\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)}-\frac{k}{n}\sum_{j=k+1}^{n} d_{\sigma_N^{-1}(j)}\right)\right]\\
=&\lambda\cdot\sum_{k=1}^{n-1}\left[\pi(n,k)\cdot\left(\frac{n-k}{n}\sum_{j=1}^{k} d_{\sigma_N^{-1}(j)}-\frac{k}{n}\sum_{j=1}^{n} d_{\sigma_N^{-1}(j)}\right)\right]\\
=&v^\theta(N),
\end{aligned}$$

where the first equality holds by (*i*) $v^{\gamma_N^\theta}(N) = v^{\gamma_\emptyset^\theta}(N) = 0$, i.e., the values of the BBHM games with sellers only (or buyers only) are zero, (*ii*) Definition 6, and (*iii*) conditioning on the cardinality of S. The second equality holds as the binomial coefficient depends on k only. The third equality holds by Lemma 6. The fourth equality holds by Lemma 1 and splitting the last sum. The fifth equality holds by some rewriting. The last equality holds by Lemma 3.

Now, let $M \subset N$. In addition, let $\widetilde{\theta} = (\widetilde{N}, \widetilde{\mu}, \widetilde{\lambda}, \widetilde{d}, \widetilde{h})$ be a new criticality situation with $\widetilde{N} = M$, $\widetilde{\lambda} = \lambda$, $\widetilde{\mu} = \mu$, $\widetilde{h} = h$ and $\widetilde{d} = (d_i)_{i\in M}$. Then

$$\begin{aligned}
&\sum_{S\subseteq N} p(\gamma_S^\theta)\cdot v^{\gamma_S^\theta}(M)\\
=&\sum_{S_1\subseteq M}\sum_{S_2\subseteq N\setminus M} p(\gamma_{S_1\cup S_2}^\theta)\cdot v^{\gamma_{S_1\cup S_2}^\theta}(M)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S_1 \subseteq M} \sum_{S_2 \subseteq N \setminus M} p(\gamma^{\theta}_{S_1 \cup S_2}) \cdot v^{\gamma^{\theta}_{S_1}}(M) \\
&= \sum_{S_1 \subseteq M} v^{\gamma^{\theta}_{S_1}}(M) \sum_{S_2 \subseteq N \setminus M} p(\gamma^{\theta}_{S_1 \cup S_2}) \\
&= \sum_{S_1 \subseteq M} v^{\gamma^{\theta}_{S_1}}(M) \sum_{S_2 \subseteq N \setminus M} \frac{\mu^{s_1+s_2} \lambda^{n-s_1-s_2}}{(\lambda+\mu)^n} \\
&= \sum_{S_1 \subseteq M} v^{\gamma^{\theta}_{S_1}}(M) \cdot \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \sum_{S_2 \subseteq N \setminus M} \frac{\mu^{s_2} \lambda^{n-m-s_2}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} v^{\gamma^{\theta}_{S_1}}(M) \cdot \frac{\mu^{S_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{\mu^{k} \lambda^{n-m-k}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} v^{\gamma^{\theta}_{S_1}}(M) \cdot \frac{\mu^{S_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \frac{(\lambda+\mu)^{n-m}}{(\lambda+\mu)^{n-m}} \\
&= \sum_{S_1 \subseteq M} \frac{\mu^{s_1} \lambda^{m-s_1}}{(\lambda+\mu)^m} \cdot v^{\gamma^{\theta}_{S_1}}(M) \\
&= \sum_{S_1 \subseteq M} p(\gamma^{\widetilde{\theta}}_{S_1}) \cdot v^{\gamma^{\widetilde{\theta}}_{S_1}}(M) \\
&= v^{\widetilde{\theta}}(M) \\
&= v^{\theta}(M),
\end{aligned}
$$

where the first equality follows by rewriting the summation as two summations. The second equality holds as sellers in $S_2 \subseteq N \setminus M$ do not influence $v^{\gamma^{\theta}_{S_1 \cup S_2}}(M)$. The third equality holds as $v^{\gamma^{\theta}_{S_1}}(M)$ does not depend on S_2. The fourth equality holds by Definition 6. The fifth equality holds as $\frac{\mu^{s_1}\lambda^{m-s_1}}{(\lambda+\mu)^m}$ does not depend on S_2. The sixth equality holds as the argument in the second summation only depends on the cardinality of S_2. The seventh equality holds by the binomium of Newton. The eighth equality holds as $\frac{(\lambda+\mu)^{n-m}}{(\lambda+\mu)^{n-m}} = 1$. The ninth equality holds as $p(\gamma^{\widetilde{\theta}}_{S_1}) = \frac{\mu^{s_1}\lambda^{m-s_1}}{(\lambda+\mu)^m}$ and $v^{\gamma^{\theta}_{S_1}}(M) = v^{\gamma^{\widetilde{\theta}}_{S_1}}(M)$ by Lemma 7. The tenth equality holds by the first part of the proof (with a game with player set M). The last equality holds by Lemma 7 again. □

Proof of Theorem 6

First, observe that for every $S \subseteq N$ with $l_S \geq 1$, i.e., with at least one trading couple, the number of orders $\omega \in \Omega$ for which a seller selects $\alpha^{\omega}_S = 1$ equals the number of orders $\omega \in \Omega$ for which a buyer selects $\alpha_S^\omega = 0$ as we consider all possible orders of the player set. As $|\Omega| = n!$, the total number of times a seller (buyer) selects is $n!/2$. Secondly, observe that for every $S \subseteq N$ with $l_S = 0$, i.e., with no trading couple, $\mathscr{L}_i(\gamma_S^\theta, \widehat{\alpha}_S) = 0$ for any $\widehat{\alpha}_s \in [0,1]$ and all $i \in N$. Now, observe that

$$\begin{aligned}
\mathscr{B}^{\widehat{\alpha}}(\theta) &= \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}\left(\gamma_S^\theta, \frac{1}{2}\right) \\
&= \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{n!} \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{n!}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{n!}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{n!} \sum_{S\subseteq N} p(\gamma_S^\theta) \sum_{\omega\in\Omega} \mathscr{L}(\gamma_S^\theta, \alpha_S^\omega) \\
&= \frac{1}{n!} \sum_{\omega\in\Omega} \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \alpha_S^\omega) \\
&= \frac{1}{n!} \sum_{\omega\in\Omega} \mathscr{B}^{\alpha^\omega}(\theta).
\end{aligned}$$

The first and last equality hold by definition. The second equality holds as $\mathscr{L}(\gamma_S^\theta, \frac{1}{2}) = \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)$ for any $S \subseteq N$. The third and fifth equality hold by some rewriting. The fourth equality is a result of the description given at the start of the proof. □

Proof of Theorem 7

First we show that *convexhull* $\left\{ \mathscr{B}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ for all $\theta \in \Theta$ and subsequently that there exists a $\theta \in \Theta$ for which $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} \not\subseteq$ *convexhull* $\left\{ \mathscr{B}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\}$ for all $\theta \in \Theta$.

Let $\theta \in \Theta$, $\widehat{\alpha}, \widehat{\alpha}' \in [0,1]^{2^N}$ and $\beta \in [0,1]$. Then observe that

$$\begin{aligned}
\beta\mathscr{B}^{\widehat{\alpha}} + (1-\beta)\mathscr{B}^{\widehat{\alpha}'} &= \beta \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \widehat{\alpha}_S) + (1-\beta) \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \widehat{\alpha}'_S) \\
&= \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \beta\widehat{\alpha}_S + (1-\beta)\widehat{\alpha}'_S) \\
&= \mathscr{B}^{\beta\widehat{\alpha}+(1-\beta)\widehat{\alpha}'}.
\end{aligned}$$

The second equality holds as $\mathscr{L}(\gamma_S^\theta, \alpha)$ is linear in α for all $S \subseteq N$. We conclude that $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$ is a convex set. In addition, observe that for all $\omega \in \Omega$ there exists an $\widehat{\alpha} \in [0,1]^{2^N}$ such that $\alpha^\omega = \widehat{\alpha}$. Hence, we conclude that *convexhull* $\left\{ \mathscr{B}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\} \subseteq \left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\}$.

Consider $\theta \in \Theta$ with $N = \{1,2,3,4,5\}$ and $d_1 > d_2 > d_3 > d_4 > d_5$. This implies that for all $S \subseteq N$ it holds that $\mathscr{L}_i(\gamma_S^\theta, 0) < \mathscr{L}_i(\gamma_S^\theta, 1)$ if $i \in S$ and active and $\mathscr{L}_i(\gamma_S^\theta, 0) > \mathscr{L}_i(\gamma_S^\theta, 1)$ if $i \in B$ and active. For all $S \subseteq N$ let $\widehat{\alpha}_S^* = 1$ if $3 \in B$ and $\widehat{\alpha}_S^* = 0$ if $3 \in S$. First, observe that

$$\mathscr{L}_3(\gamma_S^\theta, \widehat{\alpha}_S^*) \leq \mathscr{L}_3(\gamma_S^\theta, \alpha) \text{ for all } \alpha \in [0,1] \text{ and all } S \subseteq N.$$

Note that in combination with (7) it holds for all $\omega \in \Omega$ that $\mathscr{B}_3^{\widehat{\alpha}^*} < \mathscr{B}_3^{\alpha^\omega}$ if there exists an $S \subseteq N$ for which $\mathscr{L}_3\left(\gamma_S^\theta, \widehat{\alpha}_S^*\right) < \mathscr{L}_3\left(\gamma_S^\theta, \alpha_S^\omega\right)$.

Now, we show that for all $\omega \in \Omega$ there exists an $S \subseteq N$ for which $\mathscr{L}_3\left(\gamma_S^\theta, \widehat{\alpha}_S^*\right) < \mathscr{L}_3\left(\gamma_S^\theta, \alpha_S^\omega\right)$. Let $\omega \in \Omega$. We condition on $\omega(1)$.

Case 1. $\omega(1) = 5$.

For $S = \{3,5\}$ player 3 and player 5 are active sellers. Hence, $\alpha_{\{3,5\}}^\omega = 1$ while $\widehat{\alpha}_{\{3,5\}}^* = 0$ implying $\mathscr{L}_3\left(\gamma_{\{3,5\}}^\theta, \widehat{\alpha}_{\{3,5\}}^*\right) < \mathscr{L}_3\left(\gamma_{\{3,5\}}^\theta, \alpha_{\{3,5\}}^\omega\right)$.

Case 2. $\omega(1) = 4$.

Similar argument as in Case 1 with $S = \{3,4\}$.

Case 3. $\omega(1) = 3$.

Same argument as Case 1.

Case 4. $\omega(1) = 2$.

For $S = \{1,4,5\}$ player 2 and player 3 are active buyers. Hence, $\alpha_{\{1,4,5\}}^\omega = 0$ while $\widehat{\alpha}_{\{1,4,5\}}^* = 1$ implying that $\mathscr{L}_3\left(\gamma_{\{1,4,5\}}^\theta, \widehat{\alpha}_{\{1,4,5\}}^*\right) < \mathscr{L}_3\left(\gamma_{\{1,4,5\}}^\theta, \alpha_{\{1,4,5\}}^\omega\right)$.

Case 5. $\omega(1) = 1$.

Similar argument as in Case 4 with $S = \{2,4,5\}$.

Hence, for all $\omega \in \Omega$ there exists an $S \subseteq N$ for which $\mathscr{L}_3\left(\gamma_S^\theta, \widehat{\alpha}_S^*\right) < \mathscr{L}_3\left(\gamma_S^\theta, \alpha_S^\omega\right)$ and as a consequence $\mathscr{B}_3^{\widehat{\alpha}^*}(\theta) < \mathscr{B}_3^{\alpha^\omega}(\theta)$. From this we can conclude that $\left\{ \mathscr{B}^{\widehat{\alpha}} \mid \widehat{\alpha} \in [0,1]^{2^N} \right\} \not\subseteq$ *convexhull* $\left\{ \mathscr{B}^{\widehat{\alpha}^\omega} \mid \omega \in \Omega \right\}$. □

Proof of Theorem 8

Consider set $\{0,1\}^{2^{2^N}}$. Let $S \subseteq N$. The number of $\tau \in \{0,1\}^{2^{2^N}}$ for which $\tau_S = 1$ equals the number of $\tau \in \{0,1\}^{2^{2^N}}$ for which $\tau_S = 0$. As $|\{0,1\}^{2^{2^N}}| = 2^{2^N}$, the total number of $\tau \in \{0,1\}^{2^{2^N}}$ for which $\tau_S = 1$ ($\tau_S = 0$) equals $2^{2^N}/2$. Now, observe that

$$\begin{aligned}
\mathscr{B}^{\widehat{\alpha}}(\theta) &= \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}\left(\gamma_S^\theta, \frac{1}{2}\right) \\
&= \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{2^{2^N}} \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \left(\frac{2^{2^N}}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{2^{2^N}}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)\right) \\
&= \frac{1}{2^{2^N}} \sum_{S\subseteq N} p(\gamma_S^\theta) \sum_{\tau\in\{0,1\}^{2^N}} \mathscr{L}(\gamma_S^\theta, \tau_S) \\
&= \frac{1}{2^{2^N}} \sum_{\tau\in\{0,1\}^{2^N}} \sum_{S\subseteq N} p(\gamma_S^\theta) \cdot \mathscr{L}(\gamma_S^\theta, \tau_S) \\
&= \frac{1}{2^{2^N}} \sum_{\tau\in\{0,1\}^{2^N}} \mathscr{B}^{\tau}(\theta).
\end{aligned}$$

The first and last equality hold by definition. The second equality holds as $\mathscr{L}(\gamma_S^\theta, \frac{1}{2}) = \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 0) + \frac{1}{2} \cdot \mathscr{L}(\gamma_S^\theta, 1)$ for any $S \subseteq N$. The third and fifth equality hold by some rewriting. The fourth equality is a result of the description given at the start of the proof. This concludes this proof. □