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A note on "Linear programming models for a stochastic dynamic capacitated lot sizing problem"

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Abstract

Tempelmeier and Hilger (2015) study the stochastic dynamic lot sizing problem with multiple items and limited capacity. They propose a linear optimization formulation for the problem based on a piece-wise linear approximation of the non-linear functions for the expected backorders and the expected inventory position. Our work builds on Tempelmeier and Hilger (2015). We correct an erroneous derivation of the linear optimization problem and propose an improved model.

Keywords: Lot sizing, Random demand, Dynamic demand, Capacities
2010 MSC: 00-01, 99-00

1. Introduction

In this technical note we consider the stochastic dynamic capacitated lot sizing problem (SCLSP). Contrary to its deterministic counterpart, demand is assumed to be randomly distributed from a known probability distribution, in this case the Normal distribution. The problem deals with determining a production plan for $\mathcal{K}$ items ($k = 1, 2, \dots, \mathcal{K}$) over a finite horizon of T periods ($t = 1, 2, \dots, T$). All items are produced on a single resource with limited capacity C_t. We are given a forecast for each item k over the planning horizon in terms of the expected demand $\mathbb{E}[d_{kt}]$ and the related variance $Var[d_{kt}]$ per time period.

Tempelmeier and Hilger (2015) assume that the "static-uncertainty strategy" of Bookbinder and Tan (1988) applies, which means that the lot sizes as well as the periods in which to produce are determined in advance and that this plan is executed regardless of the actual demand realizations.

We show in this technical note that there is an error in the derivation of the stochastic model by Tempelmeier and Hilger (2015) and if used, would lead to incorrect production plans. In the next section we introduce the model as formulated by Tempelmeier and Hilger (2015) and while doing so we point out the error. Then we explain how this error can be corrected and propose an improved model.

2. Analysis

The deterministic counterpart of the capacitated lot sizing problem can be formulated as in Problem 1.

Problem 1 (CLSP).

$$min. \sum_{t=1}^{T}\sum_{k=1}^{\mathcal{K}}(s_k^c\gamma_{kt} + h_k^c I_{kt}) \tag{1}$$

$$s.t. \; I_{kt} = I_{k,t-1} + q_{kt} - d_{kt} \qquad \forall k,t \tag{2}$$

$$\sum_{k\in\mathcal{K}} t_k^p q_{kt} + t_k^s\gamma_{kt} \le C_t \qquad \forall k,t \tag{3}$$

$$q_{kt} \le M\gamma_{kt} \qquad \forall k,t \tag{4}$$

$$\gamma_{kt} \in \{0,1\} \qquad \forall k,t \tag{5}$$

$$0 \le q_{kt}, I_{kt} \qquad \forall k,t \tag{6}$$

In this linear optimization problem the objective is to minimize the setup cost s_k^c and the inventory holding cost h_k^c. The decision variables I_{kt}, γ_{kt} and q_{kt} represent, respectively, the inventory position, the setup decision and the production quantity. Constraint 2 represents the inventory balance equation with I_{k0} being set to some initial value. Constraint 3 limits setups and production in time period t by the available capacity C_t. Setting up production for an item k takes t_k^s amount of time and producing one item k takes t_k^p amount of time. Constraint 4 ensures that the setup variable is set to one if product k gets produced in period t (M is a sufficiently large number). Constraint 5 states that γ_{kt} is a binary decision variable. Constraint 6 ensures a lower bound on the production quantity and the inventory level.

Since demand is uncertain, a service level constraint is introduced to ensure production. This means that for the expected inventory position we obtain,

$$\mathbb{E}[I_{kt}] = Q_{kt} - \mathbb{E}[D_{kt}] + \mathcal{L}^1_{D_{kt}}(Q_{kt}) \tag{7}$$

with $\mathbb{E}[D_{kt}] = \sum_{\tau=1}^{t}\mathbb{E}[d_{kt}]$ and for the expected backlog we obtain,

$$\mathbb{E}[B^l_{kt}] = \mathcal{L}^1_{D_{kt}}(Q_{kt}) \tag{8}$$

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with $\mathcal{L}^1_{D_{kt}}(Q_{kt})$ being the first-order loss function of the random variable D_{kt}, denoting the cumulative demand, and depending on the cumulative production quantity Q_{kt}. This can be used to define the following fill-rate constraint,

$$1 - \frac{\sum_{t=1}^{T} \mathbb{E}[B_{kt}]}{\sum_{t=1}^{T} \mathbb{E}[d_{kt}]} \geq \beta^\star, \forall k \tag{9}$$

with B_{kt} denoting the backorders for product k in time period t and $\beta^\star$ being the target fill-rate. We use the fact that the expected backorders, $\mathbb{E}[B_{kt}]$, can be expressed in terms of the expected backlog, that is,

$$\mathbb{E}[B_{kt}(Q_{kt})] = \mathcal{L}^1_{D_{kt}}(Q_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(Q_{kt}) \tag{10}$$

After introduction of the expected values, Tempelmeier and Hilger (2015) derive the approximate stochastic counterpart of Problem 1 by using a piece-wise linear approximation for both functions. The functions are linearized into L line segments on the relevant interval $[u^0_{kt}; u^L_{kt}]$ where subinterval $[u^{l-1}_{kt}; u^l_{kt}]$ relates to line segment l $(1 \leq l \leq L)$. The slope associated with line segment l of the expected inventory position function for item k at time period t is as follows,

$$\Delta^l_{I_{kt}} = \bigg(\Big(u^l_{kt} - \mathbb{E}[D_{kt}] + \mathcal{L}^1_{D_{kt}}(u^l_{kt}) \Big) - \Big(u^{l-1}_{kt} - \mathbb{E}[D_{kt}] + \mathcal{L}^1_{D_{kt}}(u^{l-1}_{kt}) \Big) \bigg) \frac{1}{u^l_{kt} - u^{l-1}_{kt}} \quad \forall k,t,l \tag{11}$$

Similarly, the slope associated with line segment l of the expected backorders function for item k at time period t is as follows,

$$\Delta^l_{B_{kt}} = \Big(\mathcal{L}^1_{D_{kt}}(u^l_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(u^l_{kt}) \Big) - \Big(\mathcal{L}^1_{D_{kt}}(u^{l-1}_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(u^{l-1}_{kt}) \Big) \frac{1}{u^l_{kt} - u^{l-1}_{kt}} \quad \forall k,t,l \tag{12}$$

We now introduce a new decision variable w^l_{kt} to denote the part of the cumulative production quantity in time period t for product k and line segment l. The following equations must hold for these new decision variables,

$$w^l_{kt} = u^l_{kt} - u^{l-1}_{kt}, l = 1, 2, \ldots, l^\star - 1 \tag{13}$$

$$w^l_{kt} = Q_{kt} - u^{l-1}_{kt}, l = l^\star \tag{14}$$

$$w^l_{kt} = 0, l = l^\star + 1, l^\star + 2, \ldots, L \tag{15}$$

Tempelmeier and Hilger (2015) argue that these equations are satisfied implicitly in their model, because "the inventory function is convex, w^l_{kt} is only positive if $w^{l-1}_{kt} = u^{l-1}_{kt} - u^{l-2}_{kt}$". However, these equations are not satisfied implicitly in their model, because there is a benefit for setting those w^l_{kt}'s larger than zero where the slopes $\Delta^l_{B_{kt}}$ times w^l_{kt} contributes the most to the reduction of the expected backorders. This has to do with the fact that the backorder function found in Equation 10 is non-convex for $t \geq 2$, as stated in the next lemma.

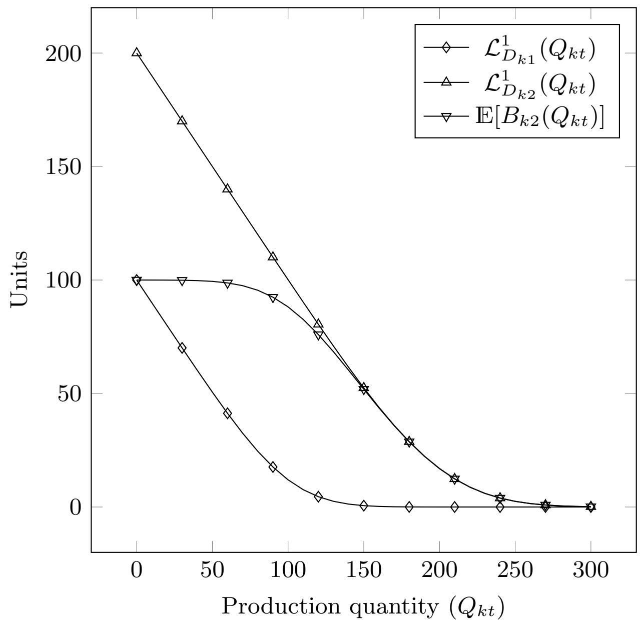

Figure 1: First-order loss functions and the backorder function for $t = 2$

Lemma 1. *The expected backorder function $\mathbb{E}[B(Q_{kt})]$ is non-convex for $t \geq 2$.*

PROOF OF LEMMA 1. See appendix.

Figure 1 further illustrates this behaviour, it shows a plot of the first-order loss functions $\mathcal{L}^1_{D_{k,1}}(Q_{kt})$, $\mathcal{L}^1_{D_{k,2}}(Q_{kt})$ and the expected backorder function $\mathbb{E}[B_{k2}(Q_{kt}]$. From this figure it becomes even more clear that we can significantly reduce the expected backorders while producing less, i.e. less w^l_{kt}'s have to be filled to their maximum, because those w^l_{kt}'s will be zero that do not add much to a reduction in the expected backorders, while those that contribute the most are filled to their maximum.

We examined the model of Tempelmeier and Hilger (2015) for selected instances of the problem. For example, we solved the problem above for one product, with the mean demand and standard deviation being, respectively, 100 and 30 for each time period over a horizon of 12 periods. Inventory holding costs are set to $h^c_k = 1$, setup costs to $s^c_k = 500$, setup time is set to zero $t^s_k = 0$, the processing time to $t^p_k = 1$ and we start with zero initial inventory $I_{k0} = 0$. Furthermore, the number of line segments in the linearization is set to $L = 18$. The goal is to achieve a target fill rate of $\beta^\star = 0.95$. The result shows that $w^1_{1,10} = w^2_{1,10} = 0$, while $w^3_{1,10} = 138.93$. This violates Equations 13, 14 and 15. The resulting production plan can be found in Table 1. Not only are the w^l_{kt}'s not filled correctly, but the resulting production plan is not in line with what is to be expected, i.e. we only produce in the first period and we produce an insufficient amount of items. This example clearly shows that our expectation is true, there are w^l_{kt}'s that are not filled correctly and this happens where the slope contributes significantly to the reduction of the expected backorders.

We can correct this error by explicitly ensuring that the w_{kt}^l's are filled sequentially using some additional constraints. We introduce a new binary decision variable $\gamma_{kt}^l \in \{0,1\}$ and add the following constraints,

$$w_{kt}^l \leq W\gamma_{kt}^l \qquad l = 2,3,\ldots,L \quad (16)$$

$$w_{kt}^{l-1} = (u_{kt}^{l-1} - u_{kt}^{l-2})\gamma_{kt}^l \qquad l = 2,3,\ldots,L \quad (17)$$

with W being a sufficiently large number.

3. Model Formulation

We correct the model proposed by Tempelmeier and Hilger (2015) and add the additional constraints formulated in Equations 16 and 17. We then arrive at a model for the approximate Stochastic Capacitated Lot Sizing Problem that we formulate in Problem 2. In this model, let $\Delta_{I_{kt}}^l$ and $\Delta_{I_{kt}}^l$ be, respectively, the expected inventory and and the expected backorder at interval end point u_{kt}^l.

Problem 2 (Approximate SCLSP).

$$min. \sum_{t=1}^{T}\sum_{k=1}^{\mathcal{K}}(s_k^c\gamma_{kt} + h_k^c[\Delta_{I_{kt}}^0 + \sum_{l=1}^{L}\Delta_{I_{kt}}^l w_{kt}^l]) \quad (18)$$

$$s.t. \sum_{k\in\mathcal{K}} t_k^p q_{kt} + t_k^s\gamma_{kt} \leq C_t \qquad \forall t \quad (19)$$

$$q_{kt} \leq M\gamma_{kt} \qquad \forall t,k \quad (20)$$

$$w_{kt}^l \leq W\lambda_{kt}^l \qquad \forall k,t,l; l \geq 2 \quad (21)$$

$$w_{kt}^{l-1} \leq (u_{kt}^{l-1} - u_{kt}^{l-2})\lambda_{kt}^l \qquad \forall k,t,l; l \geq 2 \quad (22)$$

$$\sum_{l=1}^{L} w_{k,t}^l - \sum_{l=1}^{L} w_{k,t-1}^l = q_{kt} \qquad \forall k,t \quad (23)$$

$$\sum_{l=1}^{L} w_{k,t-1}^l \leq \sum_{l=1}^{L} w_{kt}^l \qquad \forall k,t \quad (24)$$

$$\frac{\sum_{i=1}^{T}[\Delta_{B_{kt}}^0 + \sum_{l=1}^{L}\Delta_{B_{kt}}^l w_{kt}^l]}{\sum_{i=1}^{T}E[d_{ki}]} \leq 1 - \beta^\star \qquad \forall k \quad (25)$$

$$\gamma_{kt}, \lambda_{kt}^l \in \{0,1\} \qquad \forall k,t,l \quad (26)$$

$$0 \leq q_{kt} \qquad \forall k,t \quad (27)$$

If we use this model to determine the per period production quantities, then Equations 13, 14 and 15 are no longer violated and we obtain a better production plan, as Table 1 shows.

4. Conclusion

The model proposed by Tempelmeier and Hilger (2015) contains an error in the way the decision variables related to the intervals are set for each line segment. In this work we have shown how this mistake can be corrected by two additional constraints that rely on an extra binary decision variable.

References

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Appendix

Proof of Lemma 1. The expected backorder function found in Equation 10 would be convex if,

$$\frac{dB_{kt}(x)}{dx^2} \geq 0 \quad (28)$$

In order to determine if this function is convex we start with deriving its first derivative,

$$\frac{dB_{kt}(x)}{dx} = \frac{d}{dx}\Big(\mathcal{L}_{D_{kt}}^1(x) - \mathcal{L}_{D_{k,t-1}}^1(x)\Big) \quad (29)$$

$$= \frac{d}{dx}\Big(\mathcal{L}_{D_{kt}}^1(x)\Big) - \frac{d}{dx}\Big(\mathcal{L}_{D_{k,t-1}}^1(x)\Big) \quad (30)$$

$$= \frac{d}{dx}\Bigg(\sigma_{D_{kt}}\Big(\phi(z) - z(1-\Phi(zt))\Big)\Bigg) - \frac{d}{dx}\Bigg(\sigma_{D_{k,t-1}}\Big(\phi(w) - w(1-\Phi(w))\Big)\Bigg) \quad (31)$$

with $z = \frac{x-\mu_{D_{kt}}}{\sigma_{D_{kt}}}$ and $w = \frac{x-\mu_{D_{k,t-1}}}{\sigma_{D_{k,t-1}}}$. We start with deriving the first term of Equation 31,

$$\frac{d\mathcal{L}_{D_{kt}}^1(x)}{dx} = \frac{d}{dx}\Bigg(\sigma_{D_{kt}}\Big(\phi(z) - z(1-\Phi(z))\Big)\Bigg) \quad (32)$$

$$= -z\phi(z) - 1 + \Phi(z) + z\phi(z) \quad (33)$$

$$= \Phi(z) - 1 \quad (34)$$

In an analogous way we can derive the derivative of the second term of Equation 31 and combine this with Equation 34 to come to the derivative of the whole, i.e. Equation 31.

$$\frac{dB_{kt}(x)}{dx} = (\Phi(z) - 1) - (\Phi(w) - 1) \quad (35)$$

$$= \Phi(z) - \Phi(w) \quad (36)$$

We continue with determining the second derivative of B_{kt}. This can be obtained by taking the derivative of Equation 36 with respect to x. That is,

$$\frac{dB_{kt}(x)}{dx^2} = \frac{d}{dx}\Big(\Phi(z) - \Phi(w)\Big) \quad (37)$$

$$= \frac{d}{dx}\Phi(z) - \frac{d}{dx}\Phi(w) \quad (38)$$

$$= \frac{1}{\sigma_{D_t}}\phi(z) - \frac{1}{\sigma_{D_{t-1}}}\phi(w) \quad (39)$$

Table 1: A comparison between production plans.

	Time periods t											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Tempelmeier and Hilger	405.60	0	0	0	0	0	0	0	0	0	0	0
Improved model	410.09	0	0	0	439.49	0	0	0	448.30	0	0	0

For the backorder function to be convex, we need the following to hold,

$$\frac{dB_{kt}(x)}{dx^2} \geq 0 \tag{40}$$

$$\frac{1}{\sigma_{D_t}}\phi(z) - \frac{1}{\sigma_{D_{t-1}}}\phi(w) \geq 0 \tag{41}$$

$$\frac{1}{\sigma_{D_t}}\phi(z) \geq \frac{1}{\sigma_{D_{t-1}}}\phi(w) \tag{42}$$

However, a counterexample can be given to show that this inequality is not satisfied. Let the per period mean demand be 100, a standard deviation of 30, consider the case where $t = 2$ and take $x = 0$. In this case Equation 42 is not satisfied. This leads to a contradiction and from that we may conclude that the backorder function B_{kt} is non-convex for $t \geq 2$.