Base-stock policies for lost-sales models: Aggregation and asymptotics

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Abstract

This paper considers the optimization of the base-stock level for the classical periodic review lost-sales inventory system. The optimal policy for this system is not fully understood and computationally expensive to obtain. Base-stock policies for this system are asymptotically optimal as lost-sales costs approach infinity, easy to implement and prevalent in practice. Unfortunately, the state space needed to evaluate a base-stock policy exactly grows exponentially in both the lead time and the base-stock level. We show that the dynamics of this system can be aggregated into a one-dimensional state space description that grows linearly in the base-stock level only by taking a non-traditional view of the dynamics. We provide asymptotics for the transition probabilities within this single dimensional state space and show that these asymptotics have good convergence properties that are independent of the lead time under mild conditions on the demand distribution. Furthermore, we show that these asymptotics satisfy a certain flow conservation property. These results lead to a new and computationally efficient heuristic to set base-stock levels in lost-sales systems. In a numerical study we demonstrate that this approach performs better than existing heuristics with an average gap with the best base-stock policy of 0.01% across a large test-bed.

Keywords: Lost sales, base-stock policies, asymptotic results

1. Introduction

This paper studies base-stock policies for the classical lost-sales inventory problem that has been studied by Karlin and Scarf (1958), Morton (1969, 1971), van Donselaar et al. (1996), Johansen (2001), Janakiraman et al. (2007), Zipkin (2008a,b), Levi et al. (2008), Huh et al. (2009b), Goldberg et al. (2012), Bijvank et al. (2014) and Xin and Goldberg (2014). This system consists of a periodically reviewed stock point which faces stochastic i.i.d. demand. When demand in a period exceeds the on hand inventory,

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the excess is lost. Replenishment orders arrive after a lead time τ. At the end of each period, costs for lost-sales and holding inventory are charged. For such systems, we are interested in minimizing the long run average cost per period.

The structure of the optimal policy for lost-sales inventory systems with a positive replenishment lead time is still not completely understood, and the computation of optimal policies suffers from the curse of dimensionality as the state space is τ-dimensional. Goldberg et al. (2012) show that the policy to order the same quantity each period is asymptotically optimal as τ approaches infinity and Xin and Goldberg (2014) extend this result by showing that the optimality gap decays exponentially in τ. However, for moderate values of τ as encountered in practice, it is difficult to find a good policy. The only policy with a strict performance bound is the dual balancing policy proposed by Levi et al. (2008). This policy has a cost of no more than twice the optimal costs. In a numerical study, Zipkin (2008a) shows that the dual balancing policy is effective for low per unit lost-sales penalty costs, but that base-stock policies perform better in general, especially for high penalty costs. Huh et al. (2009b) show that in fact, base-stock policies are asymptotically optimal as the lost-sales penalty costs approach infinity. In fact, Bijvank et al. (2014) show that a wide range of base-stock policies is asymptotically optimal as τ approaches infinity. However, computing the best base-stock policy for a lost-sales inventory problem efficiently remains a challenge. Huh et al. (2009a), p. 398, observe that: "Although base-stock policies have been shown to perform reasonably well in lost-sales systems, finding the best base-stock policy, in general, cannot be accomplished analytically and involves simulation optimization techniques". Although the burden of optimization is alleviated by the fact that the average cost under a base-stock policy is convex in the base-stock level (Downs et al., 2001; Janakiraman and Roundy, 2004), evaluating the performance of any given base-stock policy requires either value iteration or simulation.

This paper presents asymptotic results for lost-sales systems, as do Huh et al. (2009b) and Goldberg et al. (2012), but contrary to their results, we do not focus on bounding the performance of a heuristic policy with respect to an optimal policy (although we also include such results). In-stead, we study the asymptotic dynamics of the base-stock policy for the classical lost-sales system.

We study these dynamics from a different perspective than has been done before. Our perspective is based on a relation between lost-sales and dual sourcing inventory systems that has been shown by Sheopuri et al. (2010), and results for dual sourcing inventory systems of Arts et al. (2011). Somewhat counter-intuitively, our approach involves moving from a τ-dimensional state space description to a $(\tau+1)$-dimensional state space description, where τ is the order replenishment lead time. This $(\tau+1)$-dimensional state space is the pipeline of all outstanding orders, but not the on-hand inventory. The next key idea to this approach is to aggregate this pipeline of outstanding orders into a single state variable. This is essential to lending tractability as the size of the original state space grows exponentially in both the lead time *and* the base-stock level. By contrast, the aggregated state space grows linearly in the base-stock level only.

From the distribution of this single aggregated state variable, all relevant performance measures can be computed. The distribution of this single state variable can be studied via a Markov chain. For the transition probabilities of this Markov chain, we derive asymptotic results and show that the rate of convergence for these asymptotics is at least exponential *regardless of the lead time* under mild conditions on the demand distribution. These mild conditions relate to the limiting behavior of the failure rate of the demand distribution. To show that the rate of convergence is independent of the lead time, we prove a new result on the limiting behavior of discrete failure rates of sums of random variables. We believe this result can be useful outside the present context in problems where sums of independent random variables and the failure rate play a role. Such is the case in many pricing, risk and reliability, and inventory problems.

We also show that these limiting results satisfy a type of flow conservation property. This flow conservation property relates the average size of an order entering or leaving the pipeline to the total number of items in the pipeline.

Based on all these results, we propose a simple approximation for the performance of a base-stock policy for lost-sales systems. This approximation requires the solution of $S+1$ linear equations, where S is the base-stock level. Our numerical work indicates that this approximation is very accurate. Optimization based on this approximation outperforms competing heuristics and has a cost difference with the best base-stock policy of at most 1.3% and 0.01% on average across a wide test-bed.

This paper is organized as follows. The model and notation are described in §2. In §3, we analyze the model by aggregating the state space, providing asymptotics for this aggregation. In 5 we study the rate of convergence of these asymptotics, and shows that the heuristic we suggest is asymptotically optimal as the lost sales penalty cost parameter approaches infinity under some mild distributional assumptions. In §6, we define and study flow conservation properties of approximations and verify that our approximation has this property. We consider a few small extensions in §7 and give numerical results for our approximation in §8. Concluding remarks are provided in §9.

2. Model

We consider a periodic review single stage inventory system with a replenishment lead time of τ periods ($\tau \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Periods are numbered forward in time and demand in period t is denoted D_t and $\{D_t\}_{t=0}^{\infty}$ is a sequence of non-negative i.i.d. discrete random variables with $0 < \mathbb{E}[D_t] < \infty$. We let D denote the generic single period demand random variable and we let $D^{(k)}$ denote demand over k periods. We denote the order placed in period t by Q_t and note that this order arrives in period $t+\tau$. The pipeline of orders is denoted $\mathbf{Q}_t = (Q_t, Q_{t-1}, \ldots, Q_{t-\tau})$. We let I_t denote the on-hand inventory at the beginning of period t *before* $Q_{t-\tau}$ arrives. The lost sales in period t are denoted by $L_t = (D_t - I_t + Q_{t-\tau})^+$, where $x^+ = \max(0, x)$. In each period, a holding cost of h per unit on-hand inventory *before* the arrival of an order is incurred. Lost sales are penalized with p per lost sale. The system is operated by a base-stock policy with base-stock level $S \in \mathbb{N}_0$. Thus, at the beginning of period t, an order is placed to raise the inventory position Y_t (on-hand inventory plus all outstanding orders) up to the base-stock level S:

$$Q_t = S - Y_t, \tag{1}$$

where

$$Y_t = I_t + \sum_{k=t-\tau}^{t-1} Q_k, \quad t \geq 0. \tag{2}$$

We assume without loss of generality that $I_0 \leq S$ and $Q_t = 0$ for $t = -\tau, \ldots, -1$, so that $Q_t \geq 0$ for all $t \in \mathbb{N}_0$. The random variable Q_t depends on S; to stress this, we will sometimes use the notation $Q_t(S)$. For each of the variables described, we use the subscript ∞ to denote a random variable in steady state; for instance $\mathbb{P}(I_\infty = x) = \lim_{t\to\infty} \mathbb{P}(I_t = x)$. Some care needs to be taken to ensure steady state variables do exist; Huh et al. (2009a, Theorem 3) prove that a sufficient condition for these steady state random variables to be well defined is $\mathbb{P}(D \leq S/(\tau+1)) > 0$. Most discrete distributions commonly used, such as Poisson, geometric, and (negative) binomial all satisfy this condition. Also any demand distribution with $\mathbb{P}(D = 0) > 0$ verifies this condition. Our objective will be to minimize the long run average cost per period $C(S)$ over the base-stock level S:

$$C(S) = p\mathbb{E}[L_\infty] + h\mathbb{E}[I_\infty]. \tag{3}$$

We note that this description of the problem is slightly different from most descriptions in that we account for holding costs at the beginning of a period *before* the order that is due in that period arrives, whereas we account for lost sales at the end of a period. Obviously this convention does not change the long run expected cost per period, but in the analysis, it will make the equations more transparent.

3. State space aggregation

The dynamics of I_t, L_t and Q_t are given by

$$I_{t+1} = (I_t + Q_{t-\tau} - D_t)^+, \tag{4}$$

$$L_t = (D_t - I_t - Q_{t-\tau})^+, \tag{5}$$

$$Q_{t+1} = D_t - L_t, \tag{6}$$

where $(x)^+ = \max\{0, x\}$. Define the pipeline sum, A_t, as the sum of all outstanding orders at time t, including the order that arrives in period t and the order that was placed in period t:

$$A_t = \sum_{k=t-\tau}^{t} Q_k = \mathbf{Q}_t \mathbf{e}^{\mathrm{T}}, \tag{7}$$

where $\mathbf{e}$ is the vector of all ones of length $\tau + 1$. For the pipeline sum, we have the following result.

Lemma 3.1. *The following equations hold for all $t \geq 0$*

(a) $A_t + I_t = S$

(b) $A_{t+1} = \min(S, A_t - Q_{t-\tau} + D_t)$

Proof. For (a), we can simply write using (1) and (2)

$$A_t + I_t = Q_t + \sum_{k=t-\tau}^{t-1} Q_k + I_t = S - Y_t + Y_t = S.$$

For (b), we have

$$A_{t+1} = S - I_{t+1} = S - (I_t + Q_{t-\tau} - D_t)^+ = S - (S - A_t + Q_{t-\tau} - D_t)^+ = \min(S, A_t - Q_{t-\tau} + D_t),$$

where the first equality follows from part (a), the second by substituting Equation (4), the third applying (a) again, and the final equality is easily verified by distinguishing the case $(S - A_t + Q_{t-\tau} - D_t)^+ = 0$ and $(S - A_t + Q_{t-\tau} - D_t)^+ = S - A_t + Q_{t-\tau} - D_t$. □

Finding $\mathbb{E}[A_\infty]$ gives us all the information we need to evaluate $C(S)$ because

$$\mathbb{E}[I_\infty] = S - \mathbb{E}[A_\infty] \tag{8}$$

by Lemma 3.1 (a), and

$$\mathbb{E}[L_\infty] = \mathbb{E}[D_\infty] - \mathbb{E}[Q_\infty] = \mathbb{E}[D_\infty] - \mathbb{E}[A_\infty]/(\tau+1) \tag{9}$$

by using equations (7) and (5), and so

$$C(S) = -(h + p/(\tau+1))\mathbb{E}[A_\infty] + hS + p\mathbb{E}[D_\infty]. \tag{10}$$

Finally, we note that Lemma 3.1 (b) gives us the basis for a one-dimensional Markov chain for A_t from which we can determine the distribution and mean of A_∞. This Markov chain has transition probabilities $p_{ij} = \mathbb{P}(A_{t+1} = j | A_t = i)$ that can be found by conditioning:

$$p_{ij} = \begin{cases} \lim_{t\to\infty} \sum_{k=0}^{j} \mathbb{P}(Q_{t-\tau} = i + k - j | A_t = i)\mathbb{P}(D_t = k), & \text{if } 0 \le j < S; \\ \lim_{t\to\infty} \sum_{k=0}^{i} \mathbb{P}(Q_{t-\tau} = k | A_t = i)\mathbb{P}(D_t \ge S + k - i), & \text{if } j = S. \end{cases} \tag{11}$$

Unfortunately, to evaluate $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = i | A_t = j)$, we need to evaluate the $(\tau+1)$-dimensional Markov chain $\mathbf{Q}_t$. That is,

$$\lim_{t\to\infty} \mathbb{P}\left(Q_{t-\tau} = x | A_t = y\right) = \lim_{t\to\infty} \frac{\sum_{\mathbf{q}|\mathbf{q}_{\tau+1}=x \cap \mathbf{q}\mathbf{e}^{\mathrm{T}}=y} \mathbb{P}(\mathbf{Q}_t = \mathbf{q})}{\sum_{\mathbf{q}|\mathbf{q}\mathbf{e}^{\mathrm{T}}=y} \mathbb{P}(\mathbf{Q}_t = \mathbf{q})}. \tag{12}$$

Thus, in this view of the problem, the dimension of the system just increased from τ-dimensional space to $(\tau+1)$-dimensional space and so this task suffers from the curse of dimensionality even more than finding optimal policies does. In fact, it can be shown that the state space of $\mathbf{Q}_t$ grows exponentially in both S and τ as $\binom{S+\tau+1}{S}$. (For a derivation of this result, see §A.5.) However, in the limit that $S \to \infty$, and $S = 1, 0$ we can characterize $\mathbb{P}(Q_{t-\tau} = i | A_t = j)$ and we pursue this in the next section.

4. Asymptotics

In this section, we show that as S approaches infinity and all other parameters stay constant, that

$$\mathbb{P}(Q_{t-\tau} = i | A_t = j) \to \mathbb{P}\left(D_{t-\tau-1} = i \middle| \sum_{k=t-\tau-1}^{t-1} D_k = j\right). \tag{13}$$

Furthermore, for $S = 0, 1$, (13) holds with equality in the limit that $t \to \infty$. We use these results to find an asymptotic approximation for $C(S)$. To state our results, we need some additional notation. We let $\xrightarrow{\mathcal{P}}$ denote convergence in probability.[1]

Theorem 4.1. *The following holds for all $t \geq \tau$ when everything is held constant except S:*

(a) As $S \to \infty$, $Q_{t+1} \xrightarrow{\mathcal{P}} D_t$

(b) As $S \to \infty$, $\mathbb{P}(Q_{t+1} = i) \to \mathbb{P}(D_t = i)$.

(c) As $S \to \infty$, $\mathbb{P}(Q_{t-\tau} = i | A_t = j) \to \mathbb{P}\left(D_{t-\tau-1} = i \middle| \sum_{k=t-\tau-1}^{t-1} D_k = j\right)$.

(d) For $S = 0$ and $S = 1$ and $i \leq j \leq S$,

$$\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = i | A_t = j) = \mathbb{P}\left(D_{t-\tau-1} = i \middle| \sum_{k=t-\tau-1}^{t-1} D_k = j\right).$$

Proof. First note that, by Equation (6), $Q_{t+1} \leq D_t$ with probability 1 for all $t \geq 0$. This implies in particular that $Q_{t+1} \leq_{\text{st}} D_t$, i.e., $\mathbb{P}(Q_{t+1} \leq x) \geq \mathbb{P}(D_t \leq x)$ (see Shaked and Shantikumar, 2007) and so also

$$\mathbb{P}(A_t \leq x) \geq \mathbb{P}\left(\sum_{k=t-\tau-1}^{t-1} D_k \leq x\right). \tag{14}$$

Second, we observe that $Q_{t+1} = D_t$ if and only if $L_t = 0$ which, by Equations (5) and Lemma 3.1 (a), is equivalent to the inequality

$$D_t \leq S - A_t + Q_{t-\tau}. \tag{15}$$

With this set up, we will now show that as $S \to \infty$, $Q_{t+1} \xrightarrow{\mathcal{P}} D_t$. Let $\delta \in (0, 1)$ and let S_δ satisfy $\mathbb{P}(D^{(\tau+2)} \leq S_\delta) > 1 - \delta$. (Such an $S_\delta < \infty$ exists because $\mathbb{E}[D] < \infty$ and so $\lim_{x\to\infty} \mathbb{P}\left(D^{(\tau+2)} \leq x\right) = 1$.) Now for $S \geq S_\delta$, we have

$$\begin{aligned}
\mathbb{P}(|D_t - Q_{t+1}| > 0) &= \mathbb{P}(D_t - Q_{t+1} > 0) \\
&= 1 - \mathbb{P}(D_t = Q_{t+1}) \\
&= 1 - \mathbb{P}(D_t \leq S - A_t + Q_{t-\tau}) \\
&\leq 1 - \mathbb{P}(D_t + A_t \leq S) \\
&\leq 1 - \mathbb{P}\left(D^{(\tau+2)} \leq S\right) < 1 - (1 - \delta) = \delta.
\end{aligned} \tag{16}$$

[1] A sequence of random variables X_n is said to converge in probability to X (notation $X_n \xrightarrow{\mathcal{P}} X$) if $\lim_{n\to\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0$ for all $\varepsilon > 0$.

The first equality holds because $D_t \geq Q_{t+1}$ with probability one. The second equality holds because D_t and Q_{t+1} are discrete random variables. The third equality holds because, as observed above, $Q_{t+1} = D_t$ if and only if (15) holds. The second inequality follows by substituting (14), and the final inequality follows from the fact that $S > S_\delta$. This convergence in probability implies also the convergence in distribution asserted in part (b): In the limit that S approaches infinity, $Q_{t+1} \stackrel{d}{=} D_t$ for all $t > \tau$ where $\stackrel{d}{=}$ denotes equality in distribution.

Part (c) now follows from part (b).

For part (d), the case $S = 0$ is trivial. Consider the case $S = 1$. For the condition $A_t = 0$, the result is again trivial. For the condition $A_t = 1$, we know that at time t, $Q_k = 1$ for exactly one $k \in \{t-\tau, ..., t\}$ and 0 otherwise, because $A_t \leq S$. Thus, the state space of the pipeline $\mathbf{Q}_t$, consists of the zero vector $\mathbf{0}$ and the unit vectors $\mathbf{e}_i$, for $i = 1, \cdots, \tau$, where $\mathbf{e}_i$ corresponds to the state that $Q_{t+1-i} = 1$ and $Q_k = 0$ if $k \neq t+1-i$ and $\mathbf{0}$ corresponds to an empty pipeline. The transition probabilities of $\mathbf{Q}_t$ are given by:

$$\mathbb{P}(\mathbf{Q}_{t+1} = \mathbf{x} | \mathbf{Q} = \mathbf{y}) = \begin{cases} \mathbb{P}(D = 0), & \text{if } \mathbf{x} = \mathbf{0} \text{ and } \mathbf{y} \in \{\mathbf{0}, \mathbf{e}_{\tau+1}\}; \\ \mathbb{P}(D > 0), & \text{if } \mathbf{x} = \mathbf{e}_1 \text{ and } \mathbf{y} \in \{\mathbf{0}, \mathbf{e}_{\tau+1}\}; \\ 1, & \text{if } \mathbf{x} = \mathbf{e}_{i+1} \text{ and } \mathbf{y} = \mathbf{e}_i \text{ for } i \in \{1, \ldots, \tau\}; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

It is easily verified that the stationary distribution of $\mathbf{Q}_t$ exists and satisfies $\mathbb{P}(\mathbf{Q}_\infty = \mathbf{e}_i) = \mathbb{P}(\mathbf{Q}_\infty = \mathbf{e}_{i+1})$ for $i = 1, \ldots, \tau$. From this, it follows using (12) that $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = 1 | A_t = j) = \frac{1}{\tau+1}$, and $\mathbb{P}(Q_{t-\tau} = 0 | A_t = j) = \frac{\tau}{\tau+1}$. Now, we find

$$\mathbb{P}\left(D_{t-\tau-1} = 1 \middle| \sum_{k=t-\tau-1}^{t-1} D_k = 1\right) = \frac{\mathbb{P}(D=1)\,\mathbb{P}\left(D^{(\tau)} = 0\right)}{\mathbb{P}\left(D^{(\tau+1)} = 1\right)} = \frac{\mathbb{P}(D=1)\,\mathbb{P}(D=0)^\tau}{(\tau+1)\mathbb{P}(D=1)\mathbb{P}(D=0)^\tau} = 1/(\tau+1).$$

The complement then equals $\tau/(\tau+1)$. $\square$

To state our next result, we let $\tilde{A}_\infty$ denote the random variable that results from approximating $\mathbb{P}(A_{t+1} = j | A_t = i)$ with limiting results in Theorem 4.1, i.e., $\mathbb{P}\left(\tilde{A}_\infty = x\right) = \tilde{\pi}(x)$ where $\tilde{\pi}(x)$ solves the set of linear equations

$$\tilde{\pi}(j) = \sum_{i=0}^{S} \tilde{\pi}(i)\tilde{p}_{ij}, \quad j = 0, \ldots, S-1, \qquad \sum_{i=0}^{S} \tilde{\pi}(i) = 1, \tag{18}$$

with

$$\tilde{p}_{ij} = \begin{cases} \sum_{k=0}^{j} \mathbb{P}\left(D_{t-\tau-1} = i+k-j \middle| \sum_{k=t-\tau-1}^{t-1} D_k = i\right) \mathbb{P}(D_t = k), & \text{if } j < S; \\ \sum_{k=0}^{i} \mathbb{P}\left(D_{t-\tau-1} = k \middle| \sum_{k=t-\tau-1}^{t-1} D_k = i\right) \mathbb{P}(D_t \geq S+k-i), & \text{if } j = S. \end{cases} \tag{19}$$

Furthermore, we let

$$\tilde{C}(S) = -(h + p/(\tau+1))\mathbb{E}[\tilde{A}_\infty] + hS + p\mathbb{E}\left[D_\infty\right],$$

and $\tilde{I}_\infty = S - \tilde{A}_\infty$ so that $\mathbb{P}(\tilde{I}_\infty = x) = \tilde{\pi}(S-x)$ (by Lemma 3.1 (a)).

Theorem 4.2. *If* $\mathbb{P}(D \leq S/(\tau+1)) > 0$*, then as* $S \to \infty$*,*

(a) $\tilde{p}_{ij} \longrightarrow p_{ij}$,

(b) $\tilde{\pi}(x) \longrightarrow \mathbb{P}(A_\infty = x)$,

(c) $\mathbb{E}\left[\tilde{A}_\infty\right] \longrightarrow \mathbb{E}\left[A_\infty\right]$,

(d) $\tilde{C}(S) - C(S) \longrightarrow 0$.

Furthermore we have that $\tilde{C}(1) = C(1)$ *and if* $\tau = 0$*, then* $\tilde{C}(S) = C(S)$ *for all* $S \in \mathbb{N}_0$.

Proof. Part (a) follows directly from Theorem 4.1 (c). From Huh et al. (2009a) Theorem 3, we know that under the condition $\mathbb{P}(D \leq S/(\tau+1)) > 0$, A_∞ is well defined. Consequently, $\mathbb{P}(A_\infty = x)$, $\mathbb{E}[A_\infty]$ and $C(S)$ can all be computed using only $O(S^3)$ algebraic manipulations on $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = i | A_t = j)$. Since limits are preserved under such manipulations, we obtain (b)-(d). That $\tilde{C}(1) = C(1)$ follows from Theorem 4.1 (d), and $\tilde{C}(S) = C(S)$ if $\tau = 0$ follows from observing that A_t is one-dimensional in this case and so $\tilde{A}_t = A_t$ with probability one. □

Even for rather small S, the distributions of I_∞ and A_∞ are very well approximated by the distributions of $\tilde{I}_\infty$ and $\tilde{A}_\infty$. Figure 1 illustrates this for I_∞ by showing the distribution of I_∞ as determined by simulation in conjunction with the distribution of $\tilde{I}_\infty$. The same also holds for $\tilde{C}(S)$ compared with $C(S)$ as shown in Figure 2. In §8, we report a more elaborate numerical study that shows that the approximations obtained are indeed very good across a much wider range of instances.

We conclude this section by remarking that the results above can be used to efficiently find good base-stock levels for lost-sales systems. From Downs et al. (2001), we know that $C(S)$ is convex in S, so a simple heuristic to find a good base-stock level is simply to perform a golden section search (or any other algorithm of choice) on $\tilde{C}(S)$ with the upper bound S_{UB} and lower bound S_{LB} on S given by Theorem 4 of of Huh et al. (2009b):

$$S_{UB} = \inf\left\{ y : \mathbb{P}\left(D^{(\tau+1)} \leq y\right) \geq \frac{p + h\tau}{p + h(\tau+1)} \right\}, \tag{20}$$
$$S_{LB} = \inf\left\{ y : \mathbb{P}\left(D^{(\tau+1)} \leq y\right) \geq \frac{p - h(\tau+1)}{p + h(\tau+1)} \right\}.$$

We call this heuristic the ASYMP-heuristic because it is based on asymptotic results. In the numerical section, we explore this and find that this heuristic is both accurate and fast.

5. Rates of convergence

In this section, we show that the asymptotics of the previous section have very good convergence properties under mild conditions on the demand distribution. To state our results, we introduce the hazard rate of

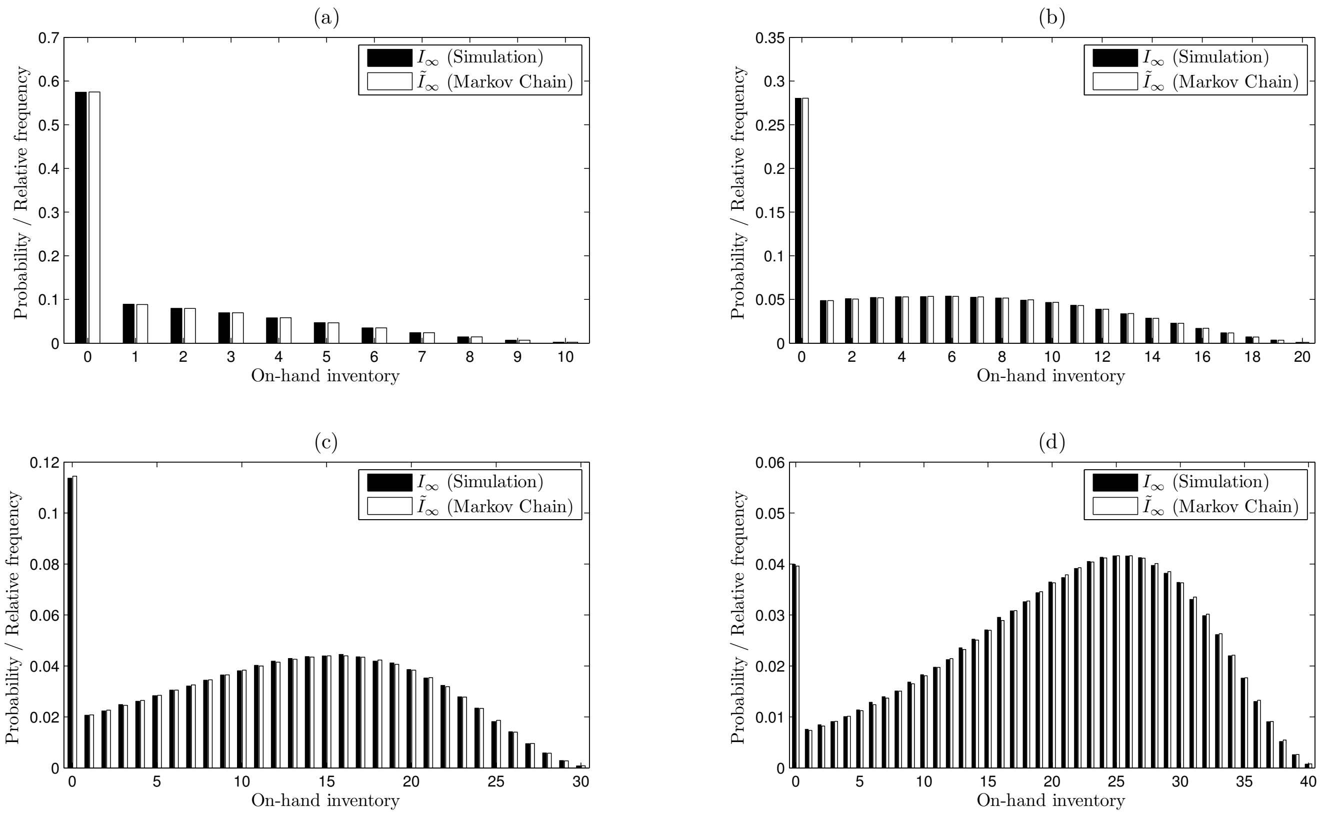

Figure 1: The distributions I_∞ as determined by simulation and of $\tilde{I}_\infty$ as determined by solving (18) for a lost-sales system with lead time $\tau = 4$ facing Geometric demand with mean 5 and base-stock levels of 10, 20, 30 and 40 in (a)-(d) respectively.

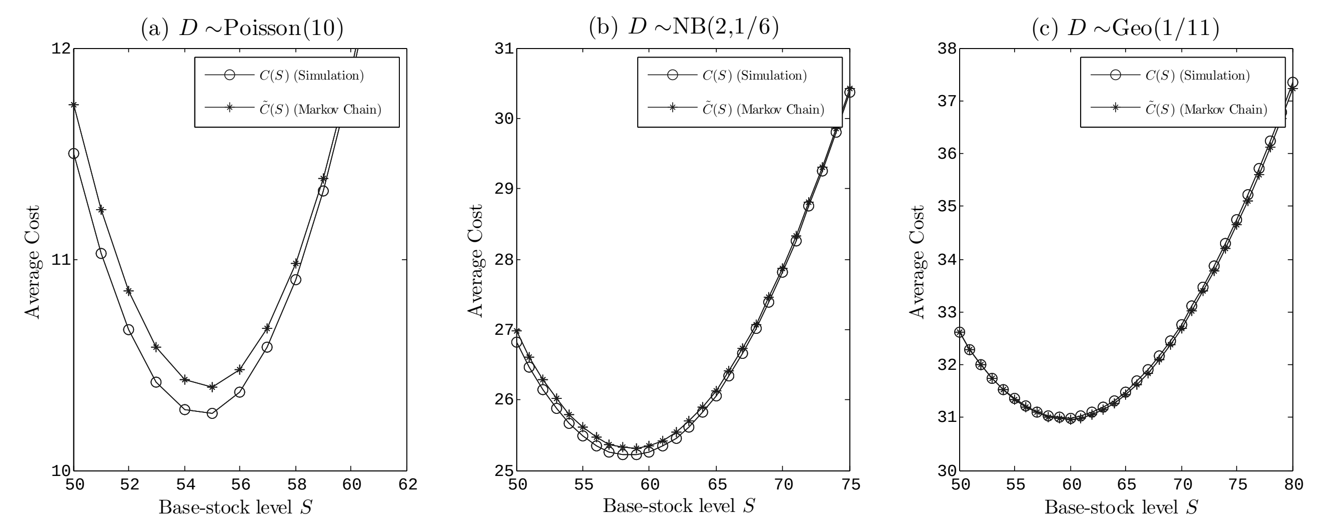

Figure 2: The true cost function $C(S)$ and the approximated cost function $\tilde{C}(S)$ for the lost-sales system with $\tau = 4$, $h = 1$, $p = 10$ for Poisson, negative binomial and geometric demand in (a)-(c) respectively. The mean demand for all these distributions is 10.

demand over k periods as

$$H^{(k)}(x) = \mathbb{P}\left(D^{(k)} = x \middle| D^{(k)} \geq x\right) = \frac{\mathbb{P}\left(D^{(k)} = x\right)}{\mathbb{P}\left(D^{(k)} \geq x\right)}.$$

We start by presenting the following lemma and proposition that relates the limit of hazard rates of sums of random variables to the limits of the hazard rates of individual random variables. We believe this proposition can be useful in different applications.

Lemma 5.1. *Let X be a non-negative discrete random variable on the integers. Then*

$$H(n) = \mathbb{P}(X = n | X \geq n) \to r$$

as $n \to \infty$ if and only if for any $m \in \mathbb{N}_0$

$$\frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > n)} \to (1 - r)^m$$

as $n \to \infty$.

The proof of this Lemma is in the appendix.

Proposition 5.2. *Let X and Y be independent discrete random variables on the integers such that $\mathbb{P}(X = n | X \geq n) \to r$ and $\mathbb{P}(Y = n | Y \geq n) \to s$ as $n \to \infty$. Then $\mathbb{P}(X + Y = n | X + Y \geq n) \to \min\{r, s\}$ as $n \to \infty$.*

Proof. Without loss of generality, assume $r \leq s$. We distinguish the following three cases: $r = 1$, $r < s \leq 1$ and $r = s < 1$. For the proof of the last two cases we use an approach similar to that of Embrechts and Goldie (1980) for a closely related property of continuous random variables.

Case $r = 1$**:** Pick $\varepsilon > 0$ and let M be such that $\mathbb{P}(X = i) \geq (1 - \varepsilon)\mathbb{P}(X \geq i)$ for $i \geq M$ and let $n > 2M$. (Such an M exists because $r = 1$.) Then we have

$$\begin{aligned}
\mathbb{P}(X + Y = n) &\geq \mathbb{P}(X + Y = n \cap X \geq M) = \textstyle\sum_{i=M}^{n} \mathbb{P}(X = i)\mathbb{P}(Y = n - i) \\
&\geq (1 - \varepsilon) \textstyle\sum_{i=M}^{n} \mathbb{P}(X \geq i)\mathbb{P}(Y = n - i) \\
&= (1 - \varepsilon)\mathbb{P}(X + Y \geq n \cap Y \leq n - M) \\
&= (1 - \varepsilon)\left[\mathbb{P}(X + Y \geq n) - \mathbb{P}(X + Y \geq n \cap Y > n - M)\right] \\
&\geq (1 - \varepsilon)\left[\mathbb{P}(X + Y \geq n) - \mathbb{P}(Y > n - M)\right].
\end{aligned}$$

The second inequality follows from our choice of M. Dividing the last result by $\mathbb{P}(X + Y \geq n)$ yields

$$1 \geq \frac{\mathbb{P}(X + Y = n)}{\mathbb{P}(X + Y \geq n)} \geq (1 - \varepsilon)\left[1 - \frac{\mathbb{P}(Y > n - M)}{\mathbb{P}(X + Y \geq n)}\right]. \tag{21}$$

Therefore it suffices to show that $\mathbb{P}(Y > n - M)/\mathbb{P}(X + Y \geq n) \to 0$ as $n \to \infty$. Observe that

$$\mathbb{P}(X + Y \geq n) \geq \mathbb{P}(X + Y \geq n \cap X \leq M + 1) \geq \mathbb{P}(Y \geq n - M - 1)\mathbb{P}(X \leq M + 1). \tag{22}$$

Therefore

$$\frac{\mathbb{P}(Y>n-M)}{\mathbb{P}(X+Y\geq n)}\leq\frac{\mathbb{P}(Y>n-M)}{\mathbb{P}(Y\geq n-M-1)\mathbb{P}(X\leq M+1)}=\frac{1-\mathbb{P}(Y=n-M-1|Y\geq n-M-1)}{\mathbb{P}(X\leq M+1)}\to 0. \tag{23}$$

The last term converges to 0 because $\mathbb{P}(Y=n-M-1|Y\geq n-M-1)\to s=1$ by assumption.

Case $r<s\leq 1$: By Lemma 5.1 we know that

$$\frac{\mathbb{P}(X>n+m)}{\mathbb{P}(X>n)}\to(1-r)^m,\qquad \frac{\mathbb{P}(Y>n+m)}{\mathbb{P}(Y>n)}\to(1-s)^m,$$

as $n\to\infty$ along the integers, for every $m\in\mathbb{N}$. This should be compared with the class of distributions called $\mathcal{L}(\gamma)$, for $\gamma\geq 0$. A random variable U is a member of $\mathcal{L}(\gamma)$ if

$$\frac{\mathbb{P}(U>x+y)}{\mathbb{P}(U>x)}\to e^{-\gamma y}$$

as $x\to\infty$ (not necessarily along the integers) for all $y>0$. Theorem 3 of Embrechts and Goldie (1980) essentially states that if $U\in\mathcal{L}(\gamma)$ and $V\in\mathcal{L}(\delta)$, then $U+V\in\mathcal{L}(\min\{\gamma,\delta\})$. Since we consider discrete random variables, X is not a member of the class $\mathcal{L}(-\ln(1-r))$, so Theorem 3 of Embrechts and Goldie (1980) does not apply directly. However, we pursue a similar line of proof for this and the next case.

Observe that

$$\begin{aligned}0&\leq\frac{\mathbb{P}(X+Y>n\cap Y>n-m)}{\mathbb{P}(X+Y>n\cap Y\leq n-m)}\\&\leq\frac{\mathbb{P}(Y>n-m)}{\mathbb{P}(X>n\cap Y\leq n-m)}\\&=\frac{\mathbb{P}(Y>n-m)}{\mathbb{P}(X>n-m)}\cdot\frac{\mathbb{P}(X>n-m)}{\mathbb{P}(X>n)}\cdot\frac{\mathbb{P}(X>n)}{\mathbb{P}(X>n)\mathbb{P}(Y\leq n-m)}\to 0\cdot(1-r)^{-m}\cdot 1=0.\end{aligned} \tag{24}$$

The last step follows from the following three observations: (1) $\frac{\mathbb{P}(Y>n-m)}{\mathbb{P}(X>n-m)}\to 0$ because $r<s$ by assumption; (2) $\frac{\mathbb{P}(X>n-m)}{\mathbb{P}(X>n)}\to(1-r)^{-m}$ by an application of Lemma 5.1; (3) $\frac{\mathbb{P}(X>n)}{\mathbb{P}(X>n)\mathbb{P}(Y\leq n-m)}=1/\mathbb{P}(Y\leq n-m)\to 1$. The above implies that

$$\frac{\mathbb{P}(X+Y>n)}{\mathbb{P}(X+Y>n\cap Y\leq n-m)}=\frac{\mathbb{P}(X+Y>n\cap Y\geq n-m)+\mathbb{P}(X+Y>n\cap Y>n-m)}{\mathbb{P}(X+Y>n\cap Y\leq n-m)}\to 1. \tag{25}$$

Let $f(n)\sim g(n)$ denote $\lim_{n\to\infty}f(n)/g(n)=1$. Then (25) can be rewritten as

$$\mathbb{P}(X+Y>n)\sim\sum_{k=0}^{n-m}\mathbb{P}(X>n-k)\mathbb{P}(Y=k), \tag{26}$$

by observing that $\mathbb{P}(X+Y>n\cap Y\leq n-m)=\sum_{k=0}^{n-m}\mathbb{P}(X>n-k)\mathbb{P}(Y=k)$. Replacing m by $m-l$ and n by $n-l$ in (26) we also have that

$$\mathbb{P}(X+Y>n-l)\sim\sum_{k=0}^{n-m}\frac{\mathbb{P}(X>n-k-l)}{\mathbb{P}(X>n-k)}\mathbb{P}(X>n-k)\mathbb{P}(Y=k). \tag{27}$$

Now define $M_X(j,l) = \sup_{n\geq j}\{\mathbb{P}(X > n-l)/\mathbb{P}(X > n)\}$ and $m_X(j,l) = \inf_{n\geq j}\{\mathbb{P}(X > n-l)/\mathbb{P}(X > n)\}$. Then the right hand side of (27) is bounded as follows:

$$m_X(n,l)\sum_{k=0}^{n-m}\mathbb{P}(X > n-k)\mathbb{P}(Y = k) \leq \sum_{k=0}^{n-m}\frac{\mathbb{P}(X > n-k-l)}{\mathbb{P}(X > n-k)}\mathbb{P}(X > n-k)\mathbb{P}(Y = k)$$
$$\leq M_X(n,l)\sum_{k=0}^{n-m}\mathbb{P}(X > n-k)\mathbb{P}(Y = k). \tag{28}$$

Now since $\lim_{n\to\infty} m_X(n,l) = \lim_{n\to\infty} M_X(n,l) = (1-r)^{-l}$ by Lemma 5.1, we obtain from (27) and (26) respectively that

$$\mathbb{P}(X+Y > n-l) \sim (1-r)^{-l}\sum_{k=0}^{n-m}\mathbb{P}(X > n-k)\mathbb{P}(Y = k) \sim (1-r)^{-l}\mathbb{P}(X+Y > n). \tag{29}$$

A last application of Lemma 5.1 completes the proof for this case.

Case $r = s < 1$**:** Observe first that for any $k > 0$ we have that

$$\mathbb{P}(X+Y > n) = \mathbb{P}(X+Y > n\cap Y \leq n-k) + \mathbb{P}(X+Y > n\cap X \leq k) + \mathbb{P}(X > k)\mathbb{P}(Y > n-k). \tag{30}$$

Define m_Y and M_Y similarly as m_X and M_X. Now for any $k > 0$

$$\begin{aligned}
\mathbb{P}(X+Y > n-l) = &\sum_{i=0}^{n-k}\frac{\mathbb{P}(X > n-i-l)}{\mathbb{P}(X > n-i)}\mathbb{P}(X > n-i)\mathbb{P}(Y = i)+\\
&\sum_{i=0}^{k-l}\frac{\mathbb{P}(Y > n-i-l)}{\mathbb{P}(Y > n-i)}\mathbb{P}(Y > n-i)\mathbb{P}(X = i)+\\
&\frac{\mathbb{P}(X > k-l)}{\mathbb{P}(X > k)}\mathbb{P}(X > k)\mathbb{P}(Y > n-k)\\
\leq\ & M_X(k,l)\sum_{i=0}^{n-k}\mathbb{P}(X > n-i)\mathbb{P}(Y = i)+\\
&M_Y(n-k+l,l)\sum_{i=0}^{k-l}\mathbb{P}(Y > n-i)\mathbb{P}(X = i) + M_X(k,l)\mathbb{P}(X > k)\mathbb{P}(Y > n-k)\\
=\ & M_X(k,l)\mathbb{P}(X+Y > n\cap Y \leq n-k)+\\
&M_Y(n-k+l,l)\mathbb{P}(X+Y > n\cap X \leq k-l)+\\
&M_X(k,l)\mathbb{P}(X+Y > n\cap X > k\cap Y > n-k)\\
\leq\ & \max\{M_X(k,l), M_Y(n-k+l,l)\}\mathbb{P}(X+Y > n).
\end{aligned}\tag{31}$$

Thus we have that

$$\limsup_{n\to\infty}\frac{\mathbb{P}(X+Y > n-l)}{\mathbb{P}(X+Y > n)} \leq \limsup_{n\to\infty}\max\{M_X(k,l), M_Y(n-k+l,l)\} = \max\{M_X(k,l),(1-r)^{-l}\}$$

for every $k \in \mathbb{N}_0$. Now let $k \to \infty$ to conclude that

$$\limsup_{n\to\infty} \frac{\mathbb{P}(X+Y>n-l)}{\mathbb{P}(X+Y>n)} \le (1-r)^{-l}. \tag{32}$$

For a lower bound on $\mathbb{P}(X+Y>n-l)$ we can write similarly

$$\begin{aligned}
\mathbb{P}(X+Y) = &\sum_{i=0}^{n-k} \frac{\mathbb{P}(X>n-i-l)}{\mathbb{P}(X>n-i)} \mathbb{P}(X>n-i)\mathbb{P}(Y=i) \\
&+ \sum_{i=0}^{k-l} \frac{\mathbb{P}(Y>n-i-l)}{\mathbb{P}(Y>n-i)} \mathbb{P})Y>n-i)\mathbb{P}(X=i) \\
&+ \frac{P(X>k-l)}{\mathbb{P}(X>k)} \mathbb{P}(X>k)\mathbb{P}(Y>n-k) \\
&\ge m_X(k,l) \sum_{i=0}^{n-k} \mathbb{P}(X>n-i)\mathbb{P}(Y=i) + \\
&\quad m_Y(n-k+l,l) \sum_{i=0}^{k-l} \mathbb{P}(Y>n-i)\mathbb{P}(X=i) + m_X(k,l)\mathbb{P}(X>k)\mathbb{P}(Y>n-k) \\
&\ge \max\{m_X(k,l), m_Y(n-k+l,l)\}\mathbb{P}(X+Y>n).
\end{aligned}$$

Now similarly we have that

$$\liminf_{n\to\infty} \frac{\mathbb{P}(X+Y>n-l)}{\mathbb{P}(X+Y>n)} \ge \liminf_{n\to\infty} \max\{m_X(k,l), m_Y(n-k+l,l)\} = \max\{m_X(k,l), (1-r)^{-l}\}$$

for all $k \in \mathbb{N}$. Let $k \to \infty$ to conclude that

$$\liminf_{n\to\infty} \frac{\mathbb{P}(X+Y>n-l)}{\mathbb{P}(X+Y>n)} \ge (1-r)^{-l}. \tag{33}$$

Combining (32) and (33) and applying Lemma 5.1 completes the proof. □

Oddly, the hazard rate properties of common discrete random variables are not found in standard literature. For the most commonly used demand models, namely Poisson, geometric, and negative binomial, we summarize results in Proposition 5.3.

Proposition 5.3. *If D is a Poisson distributed random variable, then $\mathbb{P}(D=n|D\ge n) \to 1$ as $n \to \infty$. Furthermore, if D is a negative binomially (geometrically) distributed random variable with success probability p and r required successes, then $\mathbb{P}(D=n|D\ge n) \to p$ as $n \to \infty$.*

The proof of this proposition is in the appendix. With these results, we now turn to the rate of convergence of the limits in §4.

Theorem 5.4. *If $\lim_{n\to\infty} \mathbb{P}(D=n|D\ge n) = 1-\theta \in (0,1)$, then Q_{t+1} converges to D_t in probability at least exponentially in S for any lead time τ, i.e., for any $\varepsilon \in (0, 1-\theta)$,*

$$\mathbb{P}(D_t - Q_{t+1}(S) > 0) \le O\left((\theta+\varepsilon)^S\right).$$

Furthermore, if $\lim_{n\to\infty}\mathbb{P}(D=n|n\geq n)=1$, Q_{t+1} *converges to* D_t *in probability super-exponentially in* S, *i.e., for any* $\varepsilon\in(0,1)$,

$$\mathbb{P}(D_t - Q_{t+1}(S) > 0) \leq O\left(\varepsilon^S\right).$$

Proof. From (16), we know that $\mathbb{P}(D_t - Q_{t+1}(S) > 0) \leq \mathbb{P}(D^{(\tau+2)} > S)$. Suppose that $\lim_{n\to\infty}\mathbb{P}(D = n|D\geq n) = 1-\theta\in(0,1)$. By Proposition 5.2, we have that $\lim_{x\to\infty} H^{(\tau+2)}(x) = 1-\theta\in(0,1)$. This implies that for any $\varepsilon\in(0,\theta)$, we can choose an $N\in\mathbb{N}$ such that for all $x>N$, $H^{\tau+2}(x) > 1-\theta-\varepsilon$. Now fix $C>0$ such that

$$\mathbb{P}(D^{(\tau+2)} > S) \leq C(\theta+\varepsilon)^S \tag{34}$$

for all $S\leq N$. Next observe that for $S\geq N$

$$\frac{\mathbb{P}\left(D^{(\tau+2)} > S+1\right)}{\mathbb{P}\left(D^{(\tau+2)} > S\right)} = \frac{\mathbb{P}\left(D^{(\tau+2)} > S\right) - \mathbb{P}\left(D^{(\tau+2)} = S+1\right)}{\mathbb{P}\left(D^{(\tau+2)} > S\right)} = 1 - H^{(\tau+2)}(S+1) \leq \theta+\varepsilon. \tag{35}$$

Now we proceed by induction to show that $\mathbb{P}(D^{(\tau+2)} > S) \leq C(\theta+\varepsilon)^S$ for all $S\in\mathbb{N}$. We have already verified the induction hypothesis that $\mathbb{P}(D^{\tau+2} > S) \leq C(\theta+\varepsilon)^S$ for all $S\leq N$. Suppose it holds for some $S\geq N$ and consider $S+1$:

$$\begin{aligned}\mathbb{P}\left(D^{(\tau+2)} > S+1\right) &= \frac{\mathbb{P}\left(D^{(\tau+2)} > S+1\right)}{\mathbb{P}\left(D^{(\tau+2)} > S\right)}\mathbb{P}\left(D^{(\tau+2)} > S\right)\\ &\leq (\theta+\varepsilon)\mathbb{P}\left(D^{(\tau+2)} > S\right)\\ &\leq (\theta+\varepsilon)C(\theta+\varepsilon)^S = C(\theta+\varepsilon)^{S+1}.\end{aligned}$$

The first inequality holds by using (35) and the second follows from the induction hypothesis.

The second part of the proof follows an analogous argument where $\theta=0$, and so we omit it. ☐

The results in Theorem 5.4 carry over to the rate of convergence for the limits in Theorem 4.1.

Theorem 5.5. *If* $\lim_{n\to\infty}\mathbb{P}(D=n|D\geq n) = 1-\theta\in(0,1]$, $\tilde{A}_\infty$ *converges in distribution to* A_∞ *at least exponentially fast in* S *regardless of the lead time* τ, *i.e. for any* $\varepsilon>0$ *the following hold:*

$$\begin{aligned}\mathbb{P}(A_\infty = x) &= \tilde{\pi}(x) + O\left((\theta+\varepsilon)^S\right),\\ \mathbb{E}\left[A_\infty\right] &= \mathbb{E}[\tilde{A}_\infty] + O\left((\theta+\varepsilon)^S\right),\\ C(S) &= \tilde{C}(S) + O\left((\theta+\varepsilon)^S\right).\end{aligned}$$

The proof of Theorem 5.5 is in §A.3.

Corollary 5.6. *If* D *has a Poisson distribution, then for any* $\varepsilon>0$, *it holds that* $\mathbb{P}(A_\infty = x) = \tilde{\pi}(x) + O\left(\varepsilon^S\right)$, $\mathbb{E}\left[A_\infty\right] = \mathbb{E}[\tilde{A}_\infty] + O\left(\varepsilon^S\right)$, *and* $C(S) = \tilde{C}(S) + O\left(\varepsilon^S\right)$. *If* D *has a negative binomial distribution with success probability* p, *then for any* $\varepsilon>0$, *it holds that* $\mathbb{P}(A_\infty = x) = \tilde{\pi}(x) + O\left((p+\varepsilon)^S\right)$, $\mathbb{E}\left[A_\infty\right] = \mathbb{E}[\tilde{A}_\infty] + O\left((p+\varepsilon)^S\right)$, *and* $C(S) = \tilde{C}(S) + O\left((p+\varepsilon)^S\right)$.

Since the random variable D is heavy-tailed if and only if, $\lim_{n\to\infty} \mathbb{P}(D = n|D \geq n) = 0$ (Foss et al., 2011), we have no results on the rate of convergence for heavy-tailed demand distributions. However, in the numerical section we also test our approximation for the heavy-tailed generalized Pareto distribution and find that also here the approximation performs very well.

We close this Section with an asymptotic optimality result of the heuristic we proposed in §4. We defined the ASYMP heuristic as the heuristic whose base-stock level is given by

$$S_{ASYMP} = \text{argmin}_{S\in\{S_{LB},\dots,S_{UB}\}} \tilde{C}(S), \tag{36}$$

with S_{LB} and S_{UB} given by 54. The optimal base-stock level is denoted $S^* = \text{argmin}_{S\in\{S_{LB},\dots,S_{UB}\}} C(S)$.

Theorem 5.7. *If* $\lim_{n\to\infty} \mathbb{P}(D = n \mid D \geq n) = 1 - \theta \in (0, 1)$, *then*

$$\lim_{p\to\infty} \frac{C(S_{LB})}{C(S^*)} = \lim_{p\to\infty} \frac{C(S_{UB})}{C(S^*)} = \lim_{p\to\infty} \frac{C(S_{ASYMP})}{C(S^*)} = 1.$$

The proof of this theorem is in the appendix.

6. Internal consistency: flow conservation

Our approximation relies on aggregating a pipeline of orders into a single state variable. Because A_t is originally a pipeline of orders, everything that goes in has to come out. Furthermore, everything that goes in, stays there for $\tau + 1$ periods. Thus by Little's law, we must have that

$$(\tau + 1)\mathbb{E}[Q_\infty] = \mathbb{E}[A_\infty]. \tag{37}$$

Alternatively, we might observe that $A_t = \sum_{k=t-\tau}^{t} Q_k$ also directly implies (37). In this light, we may think of (37) as expressing flow conservation: Since A_t contains $\tau + 1$ order quantities, on average the outgoing order should equal the total number of items in the pipeline divided by the length of the pipeline. Thus, an attractive property of any approximation of A_t is that it also satisfies (37) in some way. Let us make this more precise. Via (11), an approximation of $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = x|A_t = y)$ induces an approximate Markov chain for A_t. Let us denote the Markov chain induced by such an approximation $\hat{A}_t$, and let us denote the approximation for $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = x|A_t = y)$ by $\mathbb{P}(\hat{Q}_{t-\tau} = x|\hat{A}_t = y)$. Now under this approximation, the outgoing order has long run mean

$$\mathbb{E}\left[\hat{Q}_\infty\right] = \sum_{y=0}^{S} \mathbb{E}\left[\hat{Q}_{t-\tau}\middle|\hat{A}_t = y\right] \mathbb{P}\left(\hat{A}_\infty = y\right).$$

The next Proposition identifies a large class of approximations $\mathbb{P}(\hat{Q}_{t-\tau} = x|\hat{A} = y)$ that leads to an approximate chain $\hat{A}_t$ that satisfies $(\tau + 1)\mathbb{E}\left[\hat{Q}_\infty\right] = \mathbb{E}\left[\hat{A}_\infty\right]$.

Definition 1. *A Markov chain* $\hat{A}_t$ *induced by replacing* $\lim_{t\to\infty} \mathbb{P}(Q_{t-\tau} = x|A_t = y)$ *with some approximation* $\mathbb{P}(\hat{Q}_{t-\tau} = x|\hat{A}_t = y)$ *in the transition probabilities* (11) *is called* internally consistent *if it satisfies* $(\tau + 1)\mathbb{E}\left[\hat{Q}_\infty\right] = \mathbb{E}\left[\hat{A}_\infty\right]$.

With Definition 1 in place, we can state the main result of this Section.

Proposition 6.1. *Any Markov chain $\hat{A}_t$ on $0, \dots, S$ with transition probabilities $\hat{p}_{ij} = \mathbb{P}(\hat{A}_{t+1} = j | \hat{A}_t = i)$ such that*

$$\hat{p}_{ij} = \begin{cases} \sum_{k=0}^{j} \mathbb{P}\left(\hat{Q}_{t-\tau} = i + k - j \middle| \hat{A}_t = i\right) \mathbb{P}(D_t = k), & \text{if } 0 \leq j < S; \\ \sum_{k=0}^{i} \mathbb{P}\left(\hat{Q}_{t-\tau} = k \middle| \hat{A}_t = i\right) \mathbb{P}(D_t \geq S + k - i), & \text{if } j = S; \end{cases} \tag{38}$$

is internally consistent if

$$\mathbb{P}\left(\hat{Q} = x \middle| \hat{A} = y\right) = \mathbb{P}\left(X_{t-\tau} = x \middle| \sum_{k=t-\tau}^{t} X_k = y\right)$$

for some integer valued non-negative i.i.d. sequence of random variables X_t.

Proof. First observe that $\sum_{x=0}^{y} \mathbb{P}\left(X_{t-\tau} = x \middle| \sum_{k=t-\tau}^{t} X_k = y\right) = 1$ and so $\hat{A}_t$ is a Markov chain indeed.

Now we establish that $(\tau+1)\mathbb{E}\left[\hat{Q}_\infty\right] = \mathbb{E}\left[\hat{A}_\infty\right]$. Because

$$\mathbb{E}\left[X_n \middle| \sum_{k=t-\tau}^{t} X_k = y\right] = \mathbb{E}\left[X_{n+1} \middle| \sum_{k=t-\tau}^{t} X_k = y\right]$$

for any $n \in \{t-\tau, \dots, t-1\}$ and

$$\sum_{n=t-\tau}^{t} \mathbb{E}\left[X_n \middle| \sum_{k=t-\tau}^{t} X_k = y\right] = y,$$

we have that

$$\mathbb{E}\left[X_n \middle| \sum_{k=t-\tau}^{t} X_k = y\right] = y/(\tau+1). \tag{39}$$

Now for $\mathbb{E}\left[\hat{Q}_\infty\right]$ we find

$$\mathbb{E}\left[\hat{Q}_\infty\right] = \sum_{y=0}^{S} \mathbb{E}\left[X_{t-\tau} \middle| \sum_{k=t-\tau}^{t} X_k = y\right] \mathbb{P}\left(\hat{A}_\infty = y\right) = \sum_{y=0}^{S} y/(\tau+1)\mathbb{P}\left(\hat{A}_\infty = y\right) = \mathbb{E}\left[\hat{A}_\infty\right]/(\tau+1).$$

The second equality holds by substituting (39). □

Of all possible choices for X_t in Proposition 6.1, D_t is of course the most obvious because of Theorems 4.1 and 4.2.

Corollary 6.2. *$\tilde{A}_t$ is internally consistent.*

Proof. This follows from Proposition 6.1 and the assumption that D_t is a series of i.i.d. discrete non-negative random variables. □

7. Extensions

The results in the previous sections can be used for several variations of lost-sales inventory models. Below we discuss two such extensions.

7.1 General single period cost functions

Our results give approximations, not only for the moments of I_∞ and L_∞, but also for their entire distribution. Thus, a cost function that is not necessarily linear in I_t and L_t can also be accommodated. To see how the distribution of L_∞ and I_∞ can be approximated by the given results, note that by Lemma 3.1 $\mathbb{P}(I_\infty = x) = \mathbb{P}(A_\infty = S - x)$ and using Theorem 4.2, this can be approximated by $\tilde{\pi}(S - x)$. Furthermore, for the distribution of L_t we have for $x > 0$

$$\mathbb{P}(L_t = x) = \mathbb{P}\left((D_t - I_t - Q_{t-\tau})^+ = x\right) = \sum_{y=0}^{S} \mathbb{P}\left(D_t = x + y + Q_{t-\tau} | A_t = S - y\right) \mathbb{P}(I_t = y)$$
$$= \sum_{y=0}^{S} \sum_{z=0}^{S-y} \mathbb{P}(D_t = x + y + z) \mathbb{P}(I_t = y) \mathbb{P}\left(Q_{t-\tau} = z | A_t = S - y\right). \tag{40}$$

Now letting $t \to \infty$ in (40) and using the limit results in Theorems 4.1 and 4.2 to approximate, we find (again for $x > 0$):

$$\mathbb{P}(L_\infty = x) = \sum_{y=0}^{S} \sum_{z=0}^{S-y} \mathbb{P}(D_t = x + y + z) \tilde{\pi}(S - y) \mathbb{P}\left(D_{t-\tau-1} = z \middle| \textstyle\sum_{k=t-\tau-1}^{t-1} D_k = S - y\right).$$

7.2 Service level constraints

Suppose that we would like to minimize inventory holding cost while retaining the fillrate (fraction of demand not lost) above a target level $\beta \in [0, 1)$. If we choose to control this system by a base-stock policy, the objective now becomes to minimize S such that $\beta\mathbb{E}[D] \leq \mathbb{E}[A_\infty]/(\tau + 1)$. An approximate solution to this problem can be found by approximating $\mathbb{E}[A_\infty]$ by $\mathbb{E}\left[\tilde{A}_\infty\right]$.

The out-of-stock probability at the end of period, $\mathbb{P}(I_\infty = 0)$ can be approximated by $\mathbb{P}(\tilde{I}_i nfty = 0)$. Again, under a base-stock policy, this can be used to minimize inventory holding costs subject to an out-of-stock probability constraint.

8. Numerical results

In this section, we test the ASYMP-heuristic numerically by comparing it to the performance of the best base-stock policy and several other heuristics for setting base-stock levels in lost-sales inventory systems. We describe our sets of test instances and set-up in §8.1 and several heuristics in 8.2. In §8.3 we report our numerical results.

8.1 Test instances and set-up

We use and extend the test bed of Huh et al. (2009b) which is an extension of the test bed of Zipkin (2008a). (Note that the papers of Zipkin (2008a) and Huh et al. (2009b) also report the performance of the globally optimal replenishment policy.) The first and second set of instances in this test bed have Poisson and Geometric demand distributions respectively, both with mean 5 and lead times $\tau \in \{1,2,3,4\}$. The holding cost is kept constant at $h=1$ while the penalty costs $p \in \{1,4,9,19,49,99,199\}$.

The third set of test instances have Poisson demand with means ranging from 1 to 10. Holding cost is kept constant at $h=1$, $p \in \{1,4,9,14,49,99,199\}$ and $\tau \in \{2,4\}$. This set extends the set of Huh et al. (2009b) by including $\tau=4$.

The fourth set of instances has negative binomial demand with $r \in \{1,2\}$ required successes and success probability $q \in \{0.1,0.2,0.3,0.4,0.5\}$. The other parameters are as in the third set. (So here too, we add instances with $\tau=4$.)

Finally we added a fifth set of instances with heavy-tailed discretized generalized Pareto demand. Appendix B provides some details of the discretized generalized Pareto distribution. We use two distribution settings, one with shape parameter $k=0.1$ and scale parameter $\sigma=5$ and another with shape parameter $k=0.4$ and scale parameter $\sigma=10$. We again fix $h=1$, let $p \in \{1,4,9,19,49,99,199\}$ and let $\tau \in \{1,2,3,4\}$.

We compute the best base-stock levels via simulation with common random number across different base-stock levels. The results of Janakiraman and Roundy (2004) ensure that the cost function under this procedure is convex. The runlength of the simulations was set such that the halfwidth of a 99% confidence interval was less than 1% of the point estimate of the total costs. The actual performance of different heuristics is also evaluated using simulation with the same (common) random numbers.

8.2 Heuristics

The first heuristic we consider has been suggested by Huh et al. (2009b) and is asymptotically optimal as $p \to \infty$ under mild conditions on the demand distribution. The heuristic is to select the base-stock level that minimizes cost for an analogous backorder system with $p+\tau h$ as the cost per backorder per period. The resulting base-stock level is denoted S^{HS} and is the solution to a news vendor problem:

$$S^{HS} = \inf\left\{y : \mathbb{P}\left(D^{(\tau+1)} \le y\right) \ge \frac{p+\tau h}{p+(\tau+1)h}\right\}.$$

We call this heuristic the HS-heuristic. (HS stands for Huh et al. (2009b) Simple heuristic.)

Huh et al. (2009b) observe that the HS-heuristic performs quite poorly and so they suggest an improved heuristic that is also asymptotically optimal (as $p \to \infty$) and based on solving news vendor problems. This improved heuristic has base-stock level S^{HA} that satisfies

$$S^{HA} = \frac{p}{p+h} \inf\left\{y : \mathbb{P}\left(D^{(\tau+1)} \le y\right) \ge \frac{p}{p+h}\right\} + \frac{h}{p+h} \inf\left\{y : \mathbb{P}\left(D \le y\right) \ge \frac{p}{p+h}\right\}.$$

We call this heuristic the HA-heuristic. (HA stands for Huh et al. (2009b) Advanced heuristic.)

Finally we will consider an adaptation of a heuristic suggested by Bijvank and Johansen (2012), which we will denote ABJ-heuristic. The original heuristic was designed for a setting with fractional lead times,

Table 1: Average and maximum gaps with the best base-stock policy and hitrates for different heuristics for setting base-stock levels.

	Average GAP (%)				Maximum GAP (%)				Hitrate (%)			
Demand distribution	HS	HA	ABJ	ASYMP	HS	HA	ABJ	ASYMP	HS	HA	ABJ	ASYMP
Poisson mean 5	19.99	1.14	0.39	0.04	156.72	5.59	1.88	1.01	14	46	57	89
Geometric mean 5	30.67	4.61	0.00	0.00	232.25	11.22	0.02	0.00	0	0	86	100
Negative Binomial	35.71	5.28	0.01	0.00	278.13	28.21	0.13	0.17	0	0	85	92
Poisson mean 1-10	24.83	2.57	0.37	0.02	181.45	25.40	2.12	1.30	5	24	53	94
Generalized Pareto	39.23	9.05	0.00	0.00	304.58	23.32	0.07	0.04	0	0	66	73
Lead time (τ)												
1	8.87	1.71	0.11	0.04	48.17	4.38	1.88	1.01	11	25	79	89
2	19.95	3.02	0.14	0.02	125.17	18.99	2.03	1.30	4	18	73	93
3	40.06	7.47	0.06	0.00	208.70	17.55	1.13	0.04	4	4	75	75
4	43.86	5.96	0.21	0.00	304.58	28.21	2.12	0.15	0	5	62	91
Penalty cost (p)												
1	138.56	6.64	0.12	0.00	304.58	28.21	1.21	0.11	0	7	71	95
4	37.44	5.03	0.03	0.00	90.81	19.93	0.45	0.24	0	23	84	93
9	18.00	4.92	0.11	0.02	52.28	23.32	1.61	1.01	0	16	75	91
19	9.68	4.67	0.17	0.04	35.67	23.18	1.33	1.30	0	14	70	86
49	5.57	4.08	0.28	0.00	27.15	22.88	2.03	0.15	4	7	57	95
99	3.74	3.40	0.28	0.00	24.72	22.68	1.88	0.08	7	7	59	88
199	2.90	2.81	0.15	0.00	23.82	22.94	2.12	0.06	9	9	66	88
Total												
	30.84	4.51	0.16	0.01	304.58	28.21	2.12	1.30	3	12	69	91

compound Poisson demand and holding costs that are accrued continuously over time rather than at the end of a period. However, the ideas behind their heuristic can be adapted to the present setting. The idea is to apply a correction factor to an analogous backlogging system to satisfy a property that resembles flow conservation. We provide a complete derivation for the present setting in Appendix C. In short, the ABJ-heuristic chooses the base-stock level S^{ABJ} as

$$S^{ABJ} = \operatorname{argmin}_{S\in\mathbb{N}_0}\left\{hc(S)\mathbb{E}\left[\left(S-D^{(\tau+1)}\right)^+\right] + p\left(\mathbb{E}[D] - \frac{S-c(S)\mathbb{E}\left[\left(S-D^{(\tau+1)}\right)^+\right]}{\tau+1}\right)\right\}, \quad (41)$$

where $c(S)$ is a function of S given by:

$$c = \frac{S}{(\tau+1)\left(\mathbb{E}\left[\left(S-D^{(\tau)}\right)^+\right] - \mathbb{E}\left[\left(S-D^{(\tau+1)}\right)^+\right]\right) + \mathbb{E}\left[\left(S-D^{(\tau+1)}\right)^+\right]}.$$

Finally of course, there is the ASYMP-heuristic that we developed in §4. The ASYMP-heuristic has base-stock level $S^{ASYMP} = \operatorname{argmin}_{S\in\{S_{LB},\dots,S_{UB}\}} \tilde{C}(S)$.

8.3 Results

In this section, we present aggregated results about the gap with the best base-stock level and the hitrate: the percentage of instances in which a heuristic finds the best base-stock level. Arts (2013) provides details per instance in a style similar to that of Zipkin (2008a) and Huh et al. (2009b). Here we provide aggregate results in Table 1 for the average and maximum gap with the best base-stock policy and the hitrate. These results have been aggregated over the different types of demand distributions, lead times and lost-sales penalty costs.

First of all, the results show that the ASYMP-heuristic is very effective and outperforms all the other heuristics with a considerable margin. The average and maximum gap with the best base-stock level are 0.01% and 1.30% respectively and in 91% of instances, the ASYMP-heuristic found the best base-stock level. The worst case performance is for the instance with $\tau = 2$, $p = 19$ and Poisson demand with a mean of 2 per period. This is evidence that the impression given by Figures 1 and 2 holds across a very wide range of instances.

The performance of the other heuristics all degrade with lead time and improve with the penalty costs. All these heuristics are based on somehow adapting results for systems with backorders to systems with lost-sales, which explains why this happens. By contrast, the ASYMP-heuristic is not based on somehow correcting a backlogging model and so it does not suffer from the same drawbacks.

It is perhaps striking that the ASYMP-heuristic performs better for negative binomial (geometric included) and generalized Pareto demand than it does for Poisson demand, even though the theoretical convergence properties are stronger for Poisson demand; see Proposition 5.3 and Theorem 5.4. A plausible explanation for this is that for finite S, internal consistency as outlined in §6 is more instrumental in the quality of our approximation than the asymptotic results in §4. Since the ABJ-heuristic is based on correcting a backlogging model to satisfy a property that closely resembles flow conservation, this also explains why the ABJ-heuristic performs better than the HS and HA-heuristics.

We do see that the hitrate deteriorates significantly as p increases for all heuristics. This is because the best base-stock level increases with p and so the exact optimum is easier to miss.

In closing, we comment on computation times. Evaluating the best base-stock policy using value iteration is almost as difficult as determining the optimal policy. Bijvank and Johansen (2012) use a value iteration algorithm in a very similar setting and report computation times of several minutes op to several hours. We already observed that the state space required to evaluate the performance of a base-stock policy grows exponentially in both S and τ. By contrast, the state space of needed to evaluate $\tilde{C}(S)$ is linear in S only.

We determined the optimal base-stock levels with simulation and found computation times of several minutes to be the norm on a machine with 2.4 GHz Intel processor and 4GB of RAM. By contrast, all the heuristics have computation times of less than 0.01 second for all instances on the same machine.

9. Conclusion

We have presented a different view on the dynamics of a lost-sales system under a base-stock policy by focussing on the pipeline of outstanding orders. This alternate view led us to single dimensional state space description. We studied the transition probabilities within this state space using asymptotics and found that these asymptotics satisfy a type of flow conservation property. To show that the convergence of our asymptotic results is independent of the lead time, we proved a new property of the asymptotic behavior of the failure rate of discrete random variables under convolution. Based on these theoretical results, we proposed a heuristic to set base-stock levels and found that it outperforms existing heuristics and has an average and maximum gap with the best base-stock policy of 0.01% and 1.30% across a wide test-bed. Furthermore, our heuristic is computationally very efficient, the most demanding algorithmic requirement being the solution of linear equations.

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A. Proofs

A.1 Proof of Lemma 5.1

Proof. (If clause) This follows by taking $m = 1$ and observing that

$$\frac{\mathbb{P}(X > n+1)}{\mathbb{P}(X > n)} = \mathbb{P}(X > n+1|X > n) = 1 - \mathbb{P}(X = n+1|X \geq n+1) \to 1 - r.$$

(Only if) Let $A_k = \{x \in \{1, \cdots, k-1\} | \mathbb{P}(X = x) > 0\}$. The probability mass function of X can be written in terms of its hazard rate as follows:

$$\mathbb{P}(X = n) = H(n) \prod_{k \in A_n} (1 - H(k)). \tag{42}$$

Because $\mathbb{P}(X = n|X \geq n) \to r$, $\mathbb{P}(X = n) > 0$ for all sufficiently large x. Using (42), we have for sufficiently large n that

$$\frac{\mathbb{P}(X = n+1)}{\mathbb{P}(X = n)} = \frac{H(n+1)(1 - H(n)) \prod_{k \in A_n}(1 - H(k))}{H(n) \prod_{k \in A_n}(1 - H(k))} = \frac{H(n+1)}{H(n)} - H(n+1) \to 1 - r. \tag{43}$$

Using this we find

$$\frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > n)} = \mathbb{P}(X > n+m|X > n) = 1 - \mathbb{P}(X \leq n+m|X > n) \tag{44}$$

$$= 1 - \frac{\mathbb{P}(n < X \leq n+m)}{\mathbb{P}(X > n)} = 1 - \sum_{k=n+1}^{n+m} \frac{\mathbb{P}(X = k)}{\mathbb{P}(X > n)} \tag{45}$$

$$= 1 - \frac{\mathbb{P}(X = n+1)}{\mathbb{P}(X \geq n+1)} - \left(\frac{H(n+2)}{H(n+1)} - H(n+2)\right) \frac{\mathbb{P}(X = n+1)}{\mathbb{P}(X \geq n+1)} - \cdots \tag{46}$$

$$- \left(\frac{H(n+m)}{H(n+m-1)} - H(n+m)\right) \frac{\mathbb{P}(X = n+1)}{\mathbb{P}(X \geq n+1)}.$$

The fourth equality holds by substituting (43). Now as $n \to \infty$, we have by combining (46) and (43) and using that $H(n) \to r$ that

$$\frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > n)} \to 1 - r \sum_{k=0}^{m-1} (1-r)^k = 1 - r \frac{1 - (1-r)^m}{r} = (1-r)^m. \tag{47}$$

□

A.2 Proof of Proposition 5.3

Proof. Let μ denote the mean of the Poisson distributed random variable D. Then we have for $H(n) = \mathbb{P}(D = n|D \geq n)$

$$H(n) = \frac{e^{-\mu} \frac{\mu^n}{n!}}{\sum_{k=n}^{\infty} e^{-\mu} \frac{\mu^k}{k!}} = \frac{\frac{\mu^n}{n!}}{\frac{\mu^n}{n!} + \sum_{k=n+1}^{\infty} \frac{\mu^k}{k!}} = \frac{1}{1 + \frac{n!}{\mu^n} \sum_{k=n+1}^{\infty} \frac{\mu^k}{k!}}$$

$$= \frac{1}{1 + \sum_{k=1}^{\infty} \frac{\mu^k}{\prod_{j=1}^{k}(n+j)}} \geq \frac{1}{1 + \sum_{k=1}^{\infty} (\mu/n)^k}. \tag{48}$$

Now using that $\lim_{a\to 0}\sum_{k=1}^{\infty} a^k = \lim_{a\to 0} a/(1-a) = 0$, we observe that (48) implies that $\lim_{n\to\infty} H(n) \geq \lim_{n\to\infty} \frac{1}{1+\sum_{k=1}^{\infty}(\mu/n)^k} = 1$. Noting that $\mathbb{P}(D = n|D \geq n) < 1$ for all $n \in \mathbb{N}_0$, we have by the squeeze theorem that $\lim_{n\to\infty} H(n) = 1$.

If D is a negative binomial random variable, then D is the sum of several geometric random variables (for which the hazard rate is p everywhere) and so the result follows from Proposition 5.2. □

A.3 Proof of Theorem 5.5

Proof. We prove that the exponential convergence in probability of Q_{t+1} to D_t implies exponential convergence in distribution. The entire theorem then follows, as from then on, only algebraic manipulations are involved. Recall that $Q_{t+1} \leq D_t$ with probability one and so for any $a \in \mathbb{N}_0$

$$\mathbb{P}(D_t \leq a) \leq \mathbb{P}(Q_{t+1} \leq a). \tag{49}$$

Now for this same a, we have:

$$\begin{aligned}\mathbb{P}(Q_{t+1} \leq a) &= \mathbb{P}(Q_{t+1} \leq a \cap D_t \leq a) + \mathbb{P}(Q_{t+1} \leq a \cap D_t > a)\\ &\leq \mathbb{P}(D_t \leq a) + \mathbb{P}(Q_{t+1} - D_t \leq a - D_t \cap a - D_t < 0)\\ &\leq \mathbb{P}(D_t \leq a) + \mathbb{P}(D_t - Q_{t+1} > 0)\\ &= \mathbb{P}(D_t \leq a) + O\left((\theta+\varepsilon)^S\right), \end{aligned} \tag{50}$$

where (50) follows from applying Theorem 5.4. Combining (49) and (50) yields the desired result. □

A.4 Proof of Theorem 5.7

We will first present some lemma's that will facilitate our proof.

Lemma A.1. *When $\lim_{n\to\infty} H(n) = \lim_{n\to\infty} \mathbb{P}(D = n \mid D \geq n) = 1 - \theta \in (0,1)$, then for any $\varepsilon \in (0, \min(\theta, 1-\theta))$ there exists an $N \in \mathbb{N}$ and $C = \mathbb{P}(D > N) > 0$ such that*

$$C(\theta-\varepsilon)^{n-N} \leq \mathbb{P}(D > n) \leq C(\theta+\varepsilon)^{n-N} \tag{51}$$

for all $n \geq N$.

Proof. Let $\varepsilon \in (0, \min(\theta, 1-\theta))$. Pick N such that $|H(n) - 1 + \theta| < \varepsilon$ for all $n \geq N$. (Such a finite N exists by the assumption $\lim_{n\to\infty} H(n) = 1-\theta$.) Let $C = \mathbb{P}(D > N) > 0$. We will use induction. Observe that (51) holds with equality for $n = N$ by construction. Now suppose (51) holds for some $n \geq N$. We will show that it holds for $n+1$. The upper bound holds because

$$\mathbb{P}(D > n+1) = \frac{\mathbb{P}(D > n+1)}{\mathbb{P}(D > n)}\mathbb{P}(D > n) \leq (\theta+\varepsilon)\mathbb{P}(D > n) = C(\theta+\varepsilon)^{n+1-N}, \tag{52}$$

where the second inequality follows from (35) and the choice of N. The lower bound is completely analogous. □

Lemma A.2. *If* $\lim_{n\to\infty} \mathbb{P}(D = n \mid D \geq n) = 1 - \theta \in (0,1)$, *then*

$$\lim_{p\to\infty} \frac{S_{LB}}{-\log_\theta p} = \lim_{p\to\infty} \frac{S_{UB}}{-\log_\theta p} = \lim_{p\to\infty} \frac{S_{ASYMP}}{-\log_\theta p} = 1$$

Proof. Let $\varepsilon \in (0, \min(\theta, 1-\theta))$. Applying Lemma A.1 and Proposition 5.2 yields $C(\theta-\varepsilon)^{S_{LB}-N} \leq \mathbb{P}(D^{(\tau+1)} > S_{LB}) \leq C(\theta+\varepsilon)^{S_{LB}-N}$ for appropriately chosen constants N and C. Using (20), this implies that for sufficiently large p

$$S_{LB} \leq N + \log_{(\theta-\varepsilon)}\left(\frac{2C^{-1}h(\tau+1)}{p+h(\tau+1)}\right) = N + \log_{(\theta-\varepsilon)}(2C^{-1}h(\tau+1)) - \log_{(\theta-\varepsilon)}(p+h(\tau+1)) \qquad (53)$$

and

$$S_{LB} \geq N + \log_{(\theta+\varepsilon)}(2C^{-1}h(\tau+1)) - \log_{(\theta+\varepsilon)}(p+h(\tau+1)). \qquad (54)$$

Therefore, using (53), we find that

$$\lim_{p\to\infty} \frac{S_{LB}}{-\log_\theta p} \leq \lim_{p\to\infty} \frac{N + \log_{(\theta-\varepsilon)}(2C^{-1}h(\tau+1)) - \log_{(\theta-\varepsilon)}(p+h(\tau+1))}{-\log_\theta p} = \lim_{p\to\infty} \frac{\log_{(\theta-\varepsilon)}(p+h(\tau+1))}{\log_\theta p}. \qquad (55)$$

But since $\log_a x$ is continuous in a, and ε can be chosen arbitrarily in $(0, \max(\theta, 1-\theta))$ we must have (by letting $\varepsilon \downarrow 0$):

$$\lim_{p\to\infty} \frac{S_{LB}}{-\log_\theta p} \leq \lim_{p\to\infty} \frac{\log_\theta(p+h(\tau+1))}{\log_\theta(p)} = 1. \qquad (56)$$

Analogous arguments starting with (54) yield

$$\lim_{p\to\infty} \frac{S_{LB}}{-\log_\theta p} \geq 1. \qquad (57)$$

Combining (56) and (57) yields $\lim_{p\to\infty} \frac{S_{LB}}{-\log_\theta p} = 1$.

The argument to establish $\lim_{p\to\infty} \frac{S_{UB}}{-\log_\theta p} = 1$ is almost identical so we omit it. The final equality follows because $S_{LB} \leq S_{ASYMP} \leq S_{UB}$. □

Lemma A.3. *If* $\lim_{n\to\infty} \mathbb{P}(D = n \mid D \geq n) = 1 - \theta \in (0,1)$, *then*

$$\lim_{p\to\infty} \frac{C(S_{LB})}{S_{LB}} = \lim_{p\to\infty} \frac{C(S_{UB})}{S_{UB}} = h$$

Proof. By Lemma 5 of Huh et al. (2009b) we have the upper bound

$$\frac{C(S_{LB})}{S_{LB}} \geq \frac{h\mathbb{E}[(S_{LB} - D^{(\tau+1)})^+] + \frac{p}{\tau+1}\mathbb{E}[(D^{(\tau+1)} - S_{LB})^+]}{S_{LB}} \qquad (58)$$

$$= \frac{1 + \frac{p/(\tau+1)\mathbb{E}[(D^{(\tau+1)} - S_{LB})^+]}{h\mathbb{E}[(S_{LB} - D^{(\tau+1)})^+]}}{\frac{S_{LB}}{h\mathbb{E}[(S_{LB} - D^{(\tau+1)})^+]}} \qquad (59)$$

Now we will proceed to analyse $\lim_{p\to\infty}\frac{S_{LB}}{h\mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]}$ and $\frac{p/(\tau+1)\mathbb{E}[(D^{(\tau+1)}-S_{LB})^+]}{h\mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]}$. Observe that $S_{LB}\leq \mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]\leq S_{LB}-\mathbb{E}[D^{(\tau+1)}]$, so that by the squeeze theorem

$$\lim_{p\to\infty}\frac{S_{LB}}{h\mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]}=\frac{1}{h}, \tag{60}$$

because $S_{LB}\to\infty$ as $p\to\infty$ by Lemma A.2. Next we have

$$\begin{aligned}\frac{p/(\tau+1)\mathbb{E}[(D^{(\tau+1)}-S_{LB})^+]}{h\mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]}&=\frac{\frac{p}{\tau+1}\mathbb{P}(D^{(\tau+1)}>S_{LB})\mathbb{E}[D^{(\tau+1)}-S_{LB}\mid D^{(\tau+1)}>S_{LB}]}{h\mathbb{P}(D^{(\tau+1)}\leq S_{LB})\mathbb{E}[S_{LB}-D^{(\tau+1)}\mid D^{(\tau+1)}\leq S_{LB}]}\\&\leq\frac{\frac{p}{\tau+1}\frac{2h(\tau+1)}{p+h(\tau+1)}\mathbb{E}[D^{(\tau+1)}-S_{LB}\mid D^{(\tau+1)}>S_{LB}]}{h\frac{p-h(\tau+1)}{p+h(\tau+1)}\mathbb{E}[S_{LB}-D^{(\tau+1)}\mid D^{(\tau+1)}\leq S_{LB}]}\\&=\frac{2h(\tau+1)p^2+2h^2(\tau+1)^2p}{h(\tau+1)p^2-h^3(\tau+1)^3}\cdot\frac{\mathbb{E}[D^{(\tau+1)}-S_{LB}\mid D^{(\tau+1)}>S_{LB}]}{\mathbb{E}[S_{LB}-D^{(\tau+1)}\mid D^{(\tau+1)}\leq S_{LB}]}\\&\to 2\cdot 0=0\end{aligned} \tag{61}$$

as $p\to\infty$. The first inequality follows from (54). The final limit follows from observing that by Lemma A.1 and Proposition 5.2, there exists an $\varepsilon\in(0,\theta)$ such that for sufficiently large S_{LB}

$$\begin{aligned}\mathbb{E}[D^{(\tau+1)}-S_{LB}\mid D^{(\tau+1)}>S_{LB}]&=\sum_{x=0}^{\infty}\frac{\mathbb{P}(D^{(\tau+1)}=S_{LB}+x)}{\mathbb{P}(D^{(\tau+1)}>S_{LB})}\\&\leq\frac{\sum_{x=0}^{\infty}\mathbb{P}(D^{(\tau+1)}>S_LB)(\theta+\varepsilon)^x}{\mathbb{P}(D^{(\tau+1)}>S_{LB})}\\&=\frac{1}{1-\theta-\varepsilon}<\infty,\end{aligned} \tag{62}$$

and $\mathbb{E}[S_{LB}-D^{(\tau+1)}\mid D^{(\tau+1)}\leq S_{LB}]\geq S_{LB}-E[D^{(\tau+1)}]\to\infty$ as $p\to\infty$. Combining (59), (60), and (61), we conclude that $\lim_{p\to\infty}\frac{C(S_{LB})}{S_{LB}}\geq h$.

Now using the lower bound from Lemma 5 of Huh et al. (2009b) and analogous arguments we obtain

$$\frac{C(S_{LB})}{S_{LB}}\leq\frac{h\mathbb{E}[(S_{LB}-D^{(\tau+1)})^+]+(p+\tau h)\mathbb{E}[(D^{(\tau+1)}-S_{LB})^+]}{S_{LB}}\to h, \tag{63}$$

as $p\to\infty$, so that $\lim_{p\to\infty}\frac{C(S_{LB})}{S_{LB}}=h$. The proof for $\lim_{p\to\infty}C(S_{UB})/S_{UB}=1$ is analogous so we omit it. □

Now we can present the proof of Theorem 5.7.

Proof. $\lim_{p\to\infty}C(S_{UB})/C(S^*)$ is asserted in Theorem 5 of Huh et al. (2009b). Using Lemmas A.2 and A.3 this implies $\lim_{p\to\infty}\frac{C(S_{LB})/-\log_\theta p}{C(S^*)/-\log_\theta p}=1$, so that $\lim_{p\to\infty}C(S^*)/-\log_\theta p=h$. Using Lemmas A.2 and A.3 again this implies $\lim_{p\to\infty}C(S_{LB})/C(S^*)=\lim_{p\to\infty}\frac{C(S_{LB})/-\log_\theta p}{C(S^*)/-\log_\theta p}=1$. Since $C(S)$ is convex (Theorem 11 of Janakiraman and Roundy (2004)), and $S_{LB}\leq S_{ASYMP}\leq S_{UB}$, we have that $C(S_{ASYMP})\leq\max(C(S_{LB}),C(S_{UB}))$ so that also $\lim_{p\to\infty}C(S_{ASYMP})/C(S^*)=1$. □

Corollary A.4. *Consider any heuristic that finds a base-stock level S_{HEUR} such that $S_{LB} \leq S_{HEUR} \leq S_{UB}$. If $\lim_{n\to\infty} \mathbb{P}(D = n \mid D \geq n) = 1 - \theta \in (0, 1)$, then this heuristic is asymptotically optimal as $p \to \infty$, in particular*

$$\lim_{p\to\infty} C(S_{HEUR})/C(S^*) = 1.$$

Proof. Same as proof of Theorem 5.7 with S_{ASYMP} replaced by S_{HEUR}. □

A.5 Derivation of the state space size of Q_t

The size of the state space of the vector Markov chain $\mathbf{Q}_t$ is

$$\mathfrak{S}(S,\tau) = \left|\left\{\mathbf{x} \in \mathbb{N}_0^{\tau+1} | \mathbf{x}\mathbf{e}^{\mathrm{T}} \leq S\right\}\right|.$$

Now observe that $\mathfrak{S}(S,\tau)$ can be expressed recursively in τ. We have for $\tau = 0$

$$\mathfrak{S}(S,0) = \sum_{k=0}^{S} 1 = S + 1. \tag{64}$$

For $\tau = 1$ we have similarly

$$\mathfrak{S}(S,1) = \sum_{k_1=0}^{S} \sum_{k_2=0}^{S-k_1} 1 = \frac{1}{2}(S+1)(S+2), \tag{65}$$

where the second equality follows from substituting (64). We can continue such back substitution to obtain

$$\mathfrak{S}(S,2) = \sum_{k_1=0}^{S} \sum_{k_2=0}^{S-k_1} \sum_{k_3=0}^{S-k_1-k_2} 1 = \frac{1}{6}(S+1)(S+2)(S+3) \tag{66}$$

$$\mathfrak{S}(S,3) = \sum_{k_1=0}^{S} \sum_{k_2=0}^{S-k_1} \sum_{k_3=0}^{S-k_1-k_2} \sum_{k_4=0}^{S-k_1-k_2-k_3} 1 = \frac{1}{24}(S+1)(S+2)(S+3)(S+4). \tag{67}$$

It is now easy to see that

$$\mathfrak{S}(S,\tau) = \frac{1}{(\tau+1)!} \prod_{k=1}^{\tau+1} (S+k) = \frac{(S+\tau+1)!}{S!(\tau+1)!} = \binom{S+\tau+1}{S}. \tag{68}$$

Thus, $\mathfrak{S}$ grows exponentially in both τ and S.

B. The generalized Pareto distribution

A non-negative continuous random variable X is said to have a generalized Pareto distribution if

$$\mathbb{P}(X < x) = F(x) = 1 - (1 + kx/\sigma)^{-1/k}$$

for some $k > 0$ (shape parameter), $\sigma > 0$ (scale parameter) and all $x > 0$[2]. If $k < 1$, X has finite mean

$$\mathbb{E}[X] = \sigma/(1-k),$$

and if $k < 1/2$, it also has finite variance

$$\mathbf{Var}[X] = \frac{\sigma^2}{(1-k)^2(1-2k)}.$$

It is easily verified that X has a heavy-tail. If $Y = \lfloor X + 1/2 \rfloor$, then Y is said to have a discretized generalized Pareto distribution and

$$\mathbb{P}(Y = y) = F(y + 1/2) - F(y - 1/2)$$

for $y \in \mathbb{N}$.

C. Adapted Bijvank and Johansen Heuristic

The adapted Bijvank and Johansen (2012) heuristic (ABP-heuristic for short) is constructed as follows. The random variable $\hat{I}_b(S)$ should approximate the on-hand inventory at the beginning of a review period after order receipt, before demand occurs. The approximation is to apply a correction factor c to the equivalent random variable in the analogous backordering system as follows:

$$\mathbb{P}(\tilde{I}_b(S) = x) = \begin{cases} c\mathbb{P}(D^{(\tau)} = S - x), & 0 < x \leq S; \\ 1 - c\mathbb{P}(D^{(\tau)} < S), & x = 0. \end{cases}$$

Similarly, we let the random variable $\hat{I}_e(S)$ approximate the on-hand inventory at the ending of a period after demand occurs. Using the same correction factor applied to the analogous backordering system we have

$$\mathbb{P}(\tilde{I}_e(S) = x) = \begin{cases} c\mathbb{P}(D^{(\tau+1)} = S - x), & 0 < x \leq S; \\ 1 - c\mathbb{P}(D^{(\tau+1)} < S), & x = 0. \end{cases}$$

The question now becomes how the correction factor c should be determined. Since we know that $\hat{I}_b(S) - \hat{I}_e(S)$ is the order quantity under a base-stock policy, it seems reasonable to choose c to verify

$$\mathbb{E}[\hat{I}_b(S)] - \mathbb{E}[\hat{I}_e(S)] = \frac{S - \mathbb{E}[\hat{I}_e(S)]}{\tau + 1} \approx \frac{S - \mathbb{E}[I_\infty]}{\tau + 1} = \frac{\mathbb{E}[A_\infty]}{\tau + 1} = \mathbb{E}[Q_\infty]. \tag{69}$$

Note that (69) expresses something that resembles the notion of internal consistency as defined in definition 1. Solving (69) (left of the approximate equality) for c yields

$$c = \frac{S}{(\tau+1)\left(\mathbb{E}\left[\left(S - D^{(\tau)}\right)^+\right] - \mathbb{E}\left[\left(S - D^{(\tau+1)}\right)^+\right]\right) + \mathbb{E}\left[\left(S - D^{(\tau+1)}\right)^+\right]}. \tag{70}$$

[2]The generalized Pareto distribution can, and sometimes is, generalized further by introducing a location parameter and also allowing $k \leq 0$.

An approximation for the cost under a given base-stock level S is now given by:

$$\begin{aligned} \hat{C}(S) &= h\mathbb{E}\left[\hat{I}_e(S)\right] + p\left(\mathbb{E}[D] - \frac{S - \mathbb{E}\left[\hat{I}_e(S)\right]}{\tau+1}\right) \\ &= hc\mathbb{E}\left[\left(S - D^{(\tau+1)}\right)^+\right] + p\left(\mathbb{E}[D] - \frac{S - c\mathbb{E}\left[\left(S - D^{(\tau+1)}\right)^+\right]}{\tau+1}\right), \end{aligned} \tag{71}$$

where c as a function of S is given by (70). The ABJ-heuristic consists in choosing S to minimize (71).