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Abstract

We consider an environment in which several independent service providers can collaborate by pooling their critical, low-utilization resources that are subject to unavailability. We examine the allocation of the collective cost savings for such pooled situation by studying an associated cooperative game. For this game, we will prove non-emptiness of the core, present a population monotonic allocation scheme, and show convexity under some conditions. Moreover, four allocation rules will be introduced and we will investigate whether they satisfy monotonicity to availability, monotonicity to profit, situation symmetry and game symmetry. Finally, we will also investigate whether the payoff vectors resulting from those allocation rules are members of the core.

1 Introduction

In this paper, we will investigate situations in which several independent service providers keep the same type of critical, low-utilization resource that is subject to unavailability. For example, one can think of a railway setting with several contractors, each having one tamping machine. Tamping machines are critical

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resources as they repair unstable, and so unusable, railway tracks. As only a few railway tracks become unstable per year and tamping takes some hours only, utilization of tamping machines is relatively low. However, tamping machines sometimes fail, are in repair, and as a consequence are unavailable for some weeks. One can also think of a setting with several maintenance companies, each having one repairman with specific knowledge for one and the same type of highly profitable machine. Repairmen are critical resources as they repair those machines. As machines break down only a few times per year and repair takes some hours only, utilization of repairmen is relatively low. However, due to illness and vacation, repairmen may be unavailable for several days. In both examples, it can occur that there is a demand for an unavailable resource. For the railway setting, this leads to more unavailability of the railway network and as a consequence to high social costs. For the specialized repairmen setting, this leads to long(er) down time of the machine and as a consequence to lower profit. As utilization for resources is assumed to be relative low, pooling of resources may be a natural option here. Nonetheless, resource pooling may result in concerns of the service provides about their share of the total cost savings.

We will examine the allocation of the collective cost savings for such pooled situation by studying an associated cooperative game. This cooperative game, which we call a cooperative availability game, is a stylized model of reality. We assume (*i*) that resources get unavailable independently from each other and (*ii*) that one available resource can satisfy all demand if necessary. The first assumption is realistic as there is no reason to assume that a failure of a resource of a service provider would affect the failure of a resource of another service provider. The second assumption is a good approximation of reality when demand is sparse and service time per demand is not too long, i.e., when utilization is low. We will contribute in the following way. We will show that there exist allocations that cannot be improved upon by any coalition, i.e., the core is non-empty. Moreover, we present an allocation for every possible coalition such that each player's payoff increases as the coalition to which the player belongs grows larger, i.e., we present a population monotonic allocation scheme. In addition, we will present conditions that ensure that each player's marginal contribution increases as the coalition to which he or she belongs grows larger, i.e., convexity of the associated game. We will also introduce four different allocation rules and investigate whether the payoff vectors resulting from those allocation rules increase for an increasing availability and increasing profit, i.e., satisfy monotonicity to profit and monotonicity to availability. Furthermore, we will investigate whether the payoff vectors resulting from those allocation rules are the same for players that have the same profit function and availabilities, i.e., satisfy situation symmetry, and are the same for players that have the same payoff for every possible coalition, i.e., satisfy game symmetry. Finally, we will also investigate whether the payoff vectors resulting from those allocation rules are members of the core.

This paper can be positioned at the interface of cooperative game theory and operation research problems. In literature, this research area is summarized under the heading of operation research (OR) games. An overview of OR games can be found in Borm et al. (2001). They divide OR games in five categories, namely connection, routing, scheduling, production and inventory. Availability games are mostly overlapping with the last category. Recent publications in this category focus on EOQ situations (Meca et al. (2004)), economic lot sizing situations (Van den Heuvel et al. (2007)), newsvendor situations (Özen et al. (2008) and spare parts pooling situations (Karsten et al. (2012); Karsten and Basten (2014); Karsten et al. (2015)). Recently, Bachrach et al. (2012a,b, 2014) introduced and investigated a new class of operation research games, called cooperative reliability games, which comes closer to our work. Those games consider a directed network with one sink and one source, where each link is controlled by a self-interested agent. Those links are subject to failures with some fixed probability. The agents can form coalitions to obtain connectivity from the sink to the target node. A fixed reward, which is equal to the probability of achieving connectivity for that coalition, should then be divided amongst the participating agents. In particular, Bachrach et al. (2012b) focused on how to approximate the Shapley value for large networks and Bachrach et al. (2012a, 2014) focussed on when cooperative reliability games are convex and balanced. The key difference with their work is that Bachrach et al. (2012b,a,

2014) assumed that the reward obtained per coalition depends on a *single* societal profit function only, while in our model it is assumed that the reward obtained per coalition depends on the *sum* of the profit functions of all players of that coalition. Hence, results from Bachrach et al. (2012b,a, 2014) are not applicable to our situation.

The remainder of this paper is as follows. We start in Section 2 with preliminaries on cooperative game theory. Then, in Section 3 cooperative availability games will be introduced, followed by showing general properties regarding those games. In Section 4, four different allocation rules will be introduced and investigated on several properties. Finally, conclusions will be drawn in Section 5.

2 Preliminaries on cooperative game theory

In this section, we provide some basic elements of cooperative game theory. Consider a finite set $N = \{1, 2, ..., n\}$ of *players* and a function $v : 2^N \to \mathbb{R}$ called the *characteristic function*, with $v(\emptyset) = 0$. The pair (N, v) is called a *cooperative game with transferable utility*, shortly called a *game*. A subset $S \subseteq N$ is a *coalition* and $v(S)$ is the worth coalition S can achieve by itself. The worth can be transferred freely among the players. The set N is called the *grand coalition*. For a coalition $S \subseteq N$, the *subgame* (S, v_S) is the game with player set S and characteristic function v_s such that $v_s(T) = v(T)$ for all $T \subseteq S$.

A game (N, v) is called *monotonic* if the value of every coalition is at least the value of any of its subcoalitions, i.e., $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. When the value of the union of any two disjoint coalitions is larger than or equal to the sum of the values of these disjoint coalitions, a game (N, v) is called *superadditive*, i.e., $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A game (N, v) is called *convex* if the marginal contribution of any player to any coalition is less than his marginal contribution to a larger coalition, i.e., $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $S \subseteq T \subseteq N \backslash \{i\}$ and all $i \in N$.

An *allocation* for a game (N, v) is an n-dimensional vector $x \in \mathbb{R}^N$ describing the payoffs to the players, where player $i \in N$ receives x_i. An allocation is called *efficient* if $\sum_{i \in N} x_i = v(N)$. This implies that all worth is divided among the players of the grand coalition N. An allocation is called *individual rational* if $x_i \geq v(\{i\})$ for all $i \in N$ and called *stable* if no group of players has an incentive to leave the grand coalition N, i.e. $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$.

The set of efficient and individual rational allocations, called the *imputation set* of (N, v), is denoted by

$$\mathscr{I}(N,v) := \left\{ x \in \mathbb{R}^N : x_i \geq v(\{i\}) \text{ for all } i \in N \text{ and } \sum_{i \in N} x_i = v(N) \right\}.$$

The set of efficient and stable allocations, called the *core* of (N, v), is denoted by

$$\mathscr{C}(N,v) := \left\{ x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \text{ and } \sum_{i \in N} x_i = v(N) \right\}.$$

Following Bondareva (1963) and Shapley (1967), a game (N, v) is called *balanced* if the core is non-empty. If for every $S \subseteq N$, the corresponding subgame (S, v_S) is balanced, the game is called *totally balanced*.

A well known allocation rule defined on games is the *Shapley value*, proposed by Shapley (1953). The Shapley value can be described in several ways. One is to calculate a weighted average over all marginal contributions that a player can make to any possible coalition. Formally, for any game (N, v) the Shapley value is defined by

$$\Phi_i(N,v) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N|-1-|T|)!}{|N|!} v(T \cup \{i\}) - v(T) \quad \forall i \in N.$$

For any game (N, v) an *allocation scheme* $y = (y_{i,S})_{S \subseteq N, i \in S}$ specifies how to allocate the worth of every coalition. A population monotonic allocation scheme (PMAS), introduced by Sprumont (1990), is an allocation scheme $(y_{i,S})_{S \subseteq N, i \in S}$ that is *efficient*, i.e., $\sum_{i \in S} y_{i,S} = v(S)$ for all $S \subseteq N$, and *monotonic*, i.e., $y_{i,S} \leq y_{i,T}$ for all $S, T \subseteq N$ with $S \subseteq T$ and all $i \in S$. If a game (N, v) admits a PMAS y, then it is totally balanced and its allocation for the grand coalition, $(y_{i,N})_{i \in N}$, is a member of the core.

3 Model

In this section, we will introduce availability situations and define the associated games, called availability games.

3.1 Availability situations

Consider a situation with $n \in \mathbb{N}$ independent service providers, each providing the same service with a single interchangeable resource. We assume those resources to be unavailable occasionally. Let $A_i \in [0, 1]$ be the long term fraction of time that the resource of service provider i is available, i.e., the availability of service provider i, and let $1 - A_i$ be the long term fraction of time that the resource of service provider i is unavailable, i.e. unavailability of service provider i. We assume $P_i : [0, 1] \to \mathbb{R}_+$ being a non-decreasing function with $P_i(0) = 0$. For availability A_i, service provider i receives a profit of $P_i(A_i)$. We will formalize this situation by a tuple, which we will refer to as an availability situation.

Definition 1. *An availability situation is a tuple (N, A, P), where*

- $N = \{1, 2, \ldots, n\}$ *is the set of players (a player corresponds to a service provider);*
- $A = (A_i)_{i \in N}$ *is a vector of availabilities, (A_i is the availability of the service of player i);*
- $P = (P_i)_{i \in N}$ *is a vector of profit functions (P_i is the profit function of player i).*

For short, we will use θ to refer to an availability situation $\theta = (N, A, P)$ and θ' to refer to an(other) availability situation $\theta' = (N', A', P')$. Moreover, the set of availability situations is denoted by Θ.

3.2 Availability games

The service providers can protect against unavailability by pooling their resources. Here, we assume (*i*) that resources get unavailable mutually independent from each other and (*ii*) that one available resource can handle demand of all service providers if necessary. Based on those assumptions, pooling of resources works as follows. If the resource of player $i \in M$ becomes unavailable, another player in M with an available resource will help player i until his resource becomes available again. If the resource of the helping player becomes unavailable itself, another player in M with an available resource will help, and so on. Only when all resources in coalition M are unavailable, no service can be provided anymore. Hence, the availability of player i as part of coalition M becomes

$$A_i^M = 1 - \prod_{j \in M} (1 - A_j). \tag{1}$$

The profit related to player i as part of coalition M becomes $P_i(A_i^M)$ and thus the profit of coalition M becomes $\sum_{i \in M} P_i(A_i^M)$. Now, we can define a game corresponding to an availability situation θ.

Definition 2. *For any availability situation $\theta = (N, A, P)$, the game (N, v^θ) with*

$$v^\theta(M) = \sum_{i \in M} P_i \left(A_i^M \right) \tag{2}$$

for all $M \in 2^N \backslash \{\emptyset\}$ and $v^\theta(\emptyset) = 0$ is called the associated availability game.

Example 1. *Consider availability situation $\theta \in \Theta$ with $A_1 = 0.6$, $A_2 = 0.9$ and $A_3 = 0.5$ and $P_1(x) = x$, $P_2(x) = 2x$ and $P_3(x) = 7x$. In Table 1, the related availabilities and corresponding profits for (N, v^θ) are presented per coalition.* ◇

Table 1: Corresponding availabilities and profits

M	A_i^M	$v^\theta(M)$	M	A_i^M	$v^\theta(M)$
∅	0	0	{1,2}	0.96	2.88
{1}	0.60	0.60	{1,3}	0.80	6.40
{2}	0.90	1.80	{2,3}	0.95	8.55
{3}	0.50	3.50	{1,2,3}	0.98	9.80

3.3 General properties

In this section, we will present general properties for availability games. The following two Lemma's will be used frequently.

Lemma 1. *For every availability situation $\theta \in \Theta$ it holds that for any $M, K \subseteq N$ with $M \subseteq K$*

$$\prod_{i \in M}(1 - A_i) \geq \prod_{i \in K}(1 - A_i).$$

Proof : Let $\theta \in \Theta$ be an availability situation and $M, K \subseteq N$ with $M \subseteq K$. We have $0 \leq 1 - A_i \leq 1$ for all $i \in N$ and consequently

$$\prod_{i \in M}(1 - A_i) \geq \prod_{i \in M}(1 - A_i) \cdot \prod_{i \in K \setminus M}(1 - A_i) = \prod_{i \in K}(1 - A_i),$$

where the inequality uses $0 \leq \prod_{i \in S}(1 - A_i) \leq 1$ for all $S \subseteq N$. $\square$

Lemma 2. *For every availability situation $\theta \in \Theta$ with $M, K \subseteq N$, $M \subseteq K$ and $i \in M$ it holds that*

$$P_i\left(A_i^M\right) \leq P_i\left(A_i^K\right). \tag{3}$$

Proof : Let $\theta \in \Theta$ be an availability situation. Then

$$P_i\left(A_i^M\right) = P_i\left(1 - \prod_{j \in M}(1 - A_j)\right) \leq P_i\left(1 - \prod_{j \in K}(1 - A_j)\right) = P_i\left(A_i^K\right),$$

where the inequality is a result of (*i*) Lemma 1 and (*ii*) the non-decreasing property of P_i. The first and last equality follow from (1). $\square$

As a result of Lemma 2 we can now claim that availability games are monotonic.

Proposition 1. *Every availability game (N, v^θ) is monotonic.*

Proof : Let $\theta \in \Theta$ be an availability situation and (N, v^θ) be the corresponding availability game. Now, let $M, K \subseteq N$ with $M \subseteq K$. Then

$$v^\theta(M) = \sum_{i \in M} P_i(A_i^M) \leq \sum_{i \in M} P_i\left(A_i^K\right) \leq \sum_{i \in K} P_i\left(A_i^K\right) = v^\theta(K).$$

The first and last equality hold by definition. The first inequality holds by Lemma 2 and the second one holds by the combination of $P_i(0) = 0$ and the non-decreasing property of P_i. $\square$

In addition, we are able to show that every availability game (N, v^θ) is superadditive: the value of the union of disjoint coalitions is larger than or equal to the sum of the values of the disjoint subcoalitions.

Proposition 2. *Every availability game* (N, v^θ) *is superadditive.*

Proof : Let $\theta \in \Theta$ be an availability situation and (N, v^θ) be the corresponding availability game. Let $M, K \subseteq N$ with $M \cap K = \emptyset$. Then

$$\begin{aligned} v^\theta(M) + v^\theta(K) &= \sum_{i \in M} P_i\left(A_i^M\right) + \sum_{i \in K} P_i\left(A_i^K\right) \\ &\leq \sum_{i \in M} P_i\left(A_i^{M \cup K}\right) + \sum_{i \in K} P_i\left(A_i^{M \cup K}\right) \\ &= \sum_{i \in M \cup K} P_i\left(A_i^{M \cup K}\right) = v^\theta(M \cup K). \end{aligned}$$

where the inequality holds by Lemma 2. □

Superadditivity does not suffice to conclude that efficient and stable allocations exist. Following Shapley (1953), convexity of games is a sufficient condition for the existence of (an) efficient and stable allocation(s). The following example will show that availability games are not convex in general.

Example 2. *Consider the situation of Example 1. Observe that* $v(\{1,2,3\}) - v(\{2,3\}) = 9.80 - 8.55 = 1.25 < 2.90 = 6.40 - 3.50 = v(\{1,3\}) - v(\{3\})$ *and we can conclude that the game is not convex.* ◇

Despite that availability games are not convex in general, the existence of an efficient and stable allocation can still be proven.

Theorem 1. *Every availability game* (N, v^θ) *has a non-empty core.*

Proof : Let $\theta \in \Theta$ be an availability situation and (N, v^θ) be the corresponding availability game. Let $(x_i)_{i \in N}$ be the allocation with

$$x_i = P_i\left(A_i^N\right) \text{ for all } i \in N.$$

First, observe that

$$\sum_{i \in N} x_i = \sum_{i \in N} P_i\left(A_i^N\right) = v^\theta(N),$$

and thus the allocation is efficient. Secondly, observe that for any $M \subseteq N$

$$\sum_{i \in M} x_i = \sum_{i \in M} P_i\left(A_i^N\right) \geq \sum_{i \in M} P_i\left(A_i^M\right) = v^\theta(M),$$

where the inequality holds by Lemma 2. Given that $\sum_{i \in M} x_i \geq v^{\theta}(M)$ the allocation is stable as well. Hence, $(x_i)_{i \in N}$ is an efficient and stable allocation and thus always a member of the core. The core is non-empty. □

We can also claim that availability games have a population monotonic allocation scheme (PMAS).

Theorem 2. *For every availability situation $\theta \in \Theta$ the allocation scheme $(a_{i,S})_{S \subseteq N, i \in S}$, given by*

$$a_{i,S} = P_i\left(A_i^S\right) \text{ for all } i \in S \text{ and all } S \subseteq N$$

is a population monotonic allocation scheme (PMAS) for (N, v^{θ}).

Proof : Let $\theta \in \Theta$ be an availability situation. Then, observe that

$$\sum_{i \in S} a_{i,S} = \sum_{i \in S} P_i\left(A_i^S\right) = v^{\theta}(S)$$

for all $S \subseteq N$. Secondly, observe that for any $S, T \subseteq N$ with $S \subseteq T$ and $i \in S$ we have

$$a_{i,S} = P_i\left(A_i^S\right) \leq P_i\left(A_i^T\right) = a_{i,T}$$

and so $(a_{i,S})_{i \in S, S \subseteq N}$ is a PMAS. □

Following Sprumont (1990), every game with a PMAS is totally balanced. Since every availability game has a PMAS, it is totally balanced as well.

Corollary 1. *Every availability game (N, v^{θ}) is totally balanced.*

In Example 2, it is illustrated that availability games are not convex in general. However, it is of interest to investigate if there exist necessary and sufficient conditions for a class of availability situations for which convexity can be ensured. We will investigate the class of availability situations with linear profit functions, i.e., for which for every player $i \in N$, there exists a $p_i \in \mathbb{R}_+$ such that $P_i(x) = p_i x$ for all $x \in [0, 1]$. These situations will be called linear availability situations. The set of linear availability situations will be denoted by Θ^L.

Definition 3. *Let $\theta \in \Theta^L$ be a linear availability situation. Then the function $\mathscr{L}_{ij}(\theta)$ is defined by*

$$\mathscr{L}_{ij}(\theta) = A_j\left(\sum_{k \in N} p_k A_i - p_i\right) - p_j A_i \quad \text{for all } i, j \in N \text{ with } i \neq j.$$

Theorem 3. *For every linear availability situation $\theta \in \Theta^L$ with $|N| \geq 2$ and $A_i \in [0,1)$ for all $i \in N$ the corresponding game (N, v^θ) is convex if and only if*

$$\mathscr{L}_{ij}(\theta) \leq 0 \quad \textit{for all } i,j \in N \textit{ with } i \neq j.$$

Proof : See Appendix[1].

Example 3. *Consider the (linear) availability situation of Example 1. Note that $p_1 = 1$, $p_2 = 2$ and $p_3 = 7$. Then, $\mathscr{L}_{12}(\theta)$ is given by*

$$\mathscr{L}_{12}(\theta) = 0.9 \cdot (10 \cdot 0.6 - 1) - 2 \cdot 0.6 = 3.3 > 0.$$

As derived directly in Example 2, the game is indeed not convex. $\diamond$

For linear availability situations $\theta \in \Theta^L$ with $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$, Theorem 3 reduces to an easier result.

Corollary 2. *For every linear availability situation $\theta \in \Theta^L$ with $|N| \geq 2$, $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$, and $1 > A_1 \geq A_2 \geq \ldots \geq A_n$, the corresponding availability game (N, v^θ) is convex if and only if*

$$|N| A_1 A_2 - A_1 - A_2 \leq 0.$$

Proof : See Appendix.

Corollary 2 states that, under specific conditions, the corresponding availability game is convex. For example, availability games with only few players are more likely to be convex than games with many players (under the same highest and second highest availabilities). This may be due to the following effect. The additional profit player $i \in N$ generates when another player $j \in N \backslash \{i\}$ enters the coalition decreases by the size of the coalition player $i \in N$ belongs to. This effect may occur for linear availability situations $\theta \in \Theta^L$ where availabilities (and profits) are constant as well.

Corollary 3. *For every linear availability situation $\theta \in \Theta^L$ with $|N| \geq 2$ and $A_i = \overline{A} \in [0,1)$ and $p_1 \leq p_2 \leq \ldots \leq p_n$ the corresponding availability game (N, v^θ) is convex if and only if*

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{i \in N} p_i}.$$

[1]For the sake of readability, lengthy proofs are presented in the Appendix.

Proof : See Appendix.

Corollary 4. *For every linear availability situation $\theta \in \Theta^L$ with $|N| \geq 2$ and $p_i = \overline{p} \in \mathbb{R}_+$ and $A_i = \overline{A} \in [0,1)$ for all $i \in N$ the corresponding availability game (N, v^θ) is convex if and only if*

$$\overline{A} \leq \frac{2}{|N|}.$$

Proof : See Appendix.

4 Allocation Rules

In the proof of Theorem 1, an interesting allocation for every availability situation, i.e., an allocation rule, is presented. Despite that the payoff vector of this allocation rule is a core member of every availability situation, it will not necessarily satisfy any other (appealing) property. Even stronger, there may exist other allocation rules that (*i*) allocate total profit based on other criteria, (*ii*) satisfy interesting properties and (*iii*) have a payoff vector that is a core member for every availability situation as well. For that reason, we will introduce three other (interesting) allocation rules regarding availability situations. For the, in total, four allocation rules, we will investigate if they satisfy monotonicity to availability, monotonicity to profit, situation symmetry and game symmetry. Finally, we will also investigate the core membership of the payoff vectors resulting from the allocation rules.

4.1 Four allocation rules

First, we will formally introduce an allocation rule defined on availability situations.

Definition 4. *An allocation rule on availability situations is defined as a mapping γ that assigns to any availability situation $\theta \in \Theta$ a vector $\gamma(\theta) \in \mathbb{R}^N$.*

We will only pay attention to allocation rules that divide the total profit, i.e., $\sum_{i \in N} f_i(\theta) = v^\theta(N)$ for any availability situation $\theta \in \Theta$. The total profit that can be generated only depends on (*i*) the availabilities and (*ii*) the profit functions of the different players. In what follows, we will first introduce three intuitive allocation rules, each depending on the availabilities and profit functions of the different players of the corresponding availability situation. Thereafter, we will present the fourth allocation rule which is based on a well-known allocation rule for cooperative games, namely the Shapley value.

The first allocation rule (which is introduced in the proof of Theorem 1 as an allocation for every availability situation) will allocate to every player the profit, he or she generates with its *own* profit function while being part of the grand coalition. It is based on the idea that a player that generates more profit than another player under the same availability should also be rewarded more. This allocation rule, which we call Own Profit (OP), is described for any availability situation $\theta \in \Theta$ by

$$OP_i(\theta) = P_i\left(A_i^N\right) \quad \text{for all } i \in N.$$

A possible drawback of the first allocation rule is that players are not rewarded directly for the impact of their own availability (on the profit functions of others). The second allocation rule overcomes this by allocating the total profit proportional to the availabilities of the players. The idea behind this allocation rule is that the more a player is available, the more it can help others and for this it will be rewarded. Formally, for every availability situation $\theta \in \Theta$ for which there exists at least one player $j \in N$ with $A_j > 0$, this allocation rule, which we call Proportional to Availability (PA), is defined by

$$PA_i(\theta) = \frac{A_i}{\sum_{j \in N} A_j} v^\theta(N) \quad \text{for all } i \in N.$$

A possible drawback of the second allocation rule is that players are not rewarded directly for the profit generated with their own profit function while being part of the grand coalition. The third allocation rule will not overcome this (nor the other) drawback. However, it tries to find another intuitive way of dividing the profit based on the availabilities and profit functions. This allocation rule will first allocate the individual profit, i.e., the profit that every player would obtain in the individual situation, to every player. In fact, every player will be rewarded for their own availability and profit function. Then, the remaining part of the total profit will be divided proportional to the players' cost of *un*availability. Hence, the more a player is unavailable, the more it gets from the remaining part of the total profit. The idea behind this part is that the more a player is unavailable, the more (potential) profit it can generate while cooperating. For that, the player will be rewarded. Formally, for every availability situation $\theta \in \Theta$ for which there exists at least one player $j \in N$ with $A_j < 1$, this allocation rule, which we call Proportional to Unavailability Costs (PUC), is defined by

$$PUC_i(\theta) = v^\theta(\{i\}) + \frac{P_i(1) - P_i(A_i)}{\sum_{j \in N} \left[P_j(1) - P_j(A_j)\right]} \left(v^\theta(N) - \sum_{j \in N} v^\theta(\{j\}) \right) \text{ for all } i \in N.$$

The last allocation rule that will be introduced is the Shapley value. The Shapley value (Shapley (1953)) is a well-known (and accepted) allocation rule for cooperative games. It is the only one that satisfies the efficiency, monotonicity, symmetry and dummy property simultaneously. We will define the Shapley value (SV) for every availability situation $\theta \in \Theta$ by

$$\text{SV}_i(\theta) = \Phi_i(N, v^\theta) \quad \text{for all } i \in N.$$

4.2 Properties of allocation rules

In this section we will investigate whether the allocation rules satisfy intuitive properties as monotonicity to availability, monotonicity to profit, situation symmetry and game symmetry. Finally, we will also investigate whether the payoff vectors resulting from the allocation rules are core members.

4.2.1 Monotonicity to availability

Suppose the availability of a player increases. Then, this specific player is able to generate more profit. Moreover, as the total availability increases, other players can generate more profit as well. Hence, it is natural to assume that players do not expect decreases in their allocations. We will investigate whether the allocation rules will allocate to all players not less when the availability of any player increases., i.e., satisfy monotonicity to availability.

Definition 5. *An allocation rule γ satisfies monotonicity to availability on $D \subseteq \Theta$ if for any two availability situations $\theta, \theta' \in D$, where θ and θ' coincide except for the availability of player j with $A_j \leq A'_j$*

$$\gamma_i(\theta) \leq \gamma_i(\theta') \quad \text{for all } i \in N.$$

The following example will show that allocation rules *PA*, *PUC* and *SV* do not satisfy monotonicity to availability on Θ.

Example 4. *Consider availability situation $\theta \in \Theta$ with $N = \{1,2,3\}$, $A_1 = 0.5$, $A_2 = 0.5$, $A_3 = 0.5$ and*

$$P_1(x) = P_2(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \text{if} \quad \frac{1}{2} < x < 1 \\ 1 & \text{if} \quad x = 1, \end{cases} \qquad P_3(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

Moreover, consider situation $\theta' \in \Theta$, which coincides with θ except that $A'_3 = 0.75$. In Table 2, the four allocations regarding those two situations θ and θ' are depicted for all three players.

Table 2: Allocations for availability game

	i	OP_i	PA_i	PUC_i	SV_i		i	OP_i	PA_i	PUC_i	SV_i
	1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{12}$		1	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{1}{2}$
θ	2	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{12}$	θ'	2	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{1}{2}$
	3	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{10}{12}$		3	1	$\frac{6}{7}$	1	1

Allocation rules PA, PUC and SV do not satisfy monotonicity to availability, since

$$PA_2(\theta) = \frac{2}{3} > \frac{4}{7} = PA_2(\theta'),$$
$$PU_2(\theta) = \frac{2}{3} > \frac{1}{2} = PU_2(\theta'),$$
$$SV_2(\theta) = \frac{7}{12} > \frac{1}{2} = SV_3(\theta').$$

Note that this example can also be constructed with continious profit functions. ◇

Theorem 4. *Allocation rule OP satisfies monotonicity to availability on* Θ.

Proof : See Appendix.

For linear availability situations, we obtain different results regarding monotonicity to availability. The following example will be used to show that allocation rules PA and SV do not satisfy monotonicity to availability on Θ^L.

Example 5. *Consider the (linear) availability situation* $\theta \in \Theta^L$ *of Example 1. Moreover, consider situation* $\theta' \in \Theta^L$, *which coincides with* θ *except that* $A_1' = 0.8$ *now. In Table 3, the four allocations regarding those two situations* θ *and* θ' *are depicted for all three players. All numbers are rounded to two decimals.*

Table 3: Allocations for availability game

	i	OP_i	PA_i	PUC_i	SV_i		i	OP_i	PA_i	PUC_i	SV_i
	1	0.98	2.94	0.98	1.28		1	0.99	3.60	0.99	1.52
θ	2	1.96	4.41	1.99	2.95	θ'	2	1.98	4.05	2.00	2.70
	3	6.86	2.45	6.83	5.57		3	6.93	2.25	6.91	5.68

Allocation rules PA and SV do not satisfy monotonicity to availability, since

$$PA_2(\theta) > 4.40 > 4.06 > PA_2(\theta'),$$
$$SV_2(\theta) > 2.94 > 2.71 > SV_2(\theta'). \qquad \diamond$$

Theorem 5. *Allocation rules OP and PUC satisfy monotonicity to availability on* Θ^L.

Proof : See Appendix.

4.2.2 Monotonicity to profit

Suppose the profit function of a player changes such that the outcome of the difference between the old and the new profit function increases for an increasing availability. Then, this specific player is able to generate more profit. Despite that the other players will not generate more profit themselves, they are responsible (in terms of availability) for the (extra) profit of the specific player as well. Hence, it is natural to assume that players do not expect decreases in their allocations. We will investigate whether the allocation rules will allocate to all players not less when the difference between the outcome of the new and old profit function of a specific player is non-decreasing in the availability, i.e., satisfy monotonicity to profit.

Definition 6. *An allocation rule γ satisfies monotonicity to profit on $D \subseteq \Theta$ if for any two availability situations $\theta, \theta' \in D$, where θ and θ' coincide except for the profit of player j with $P'_j(x) - P_j(x)$ non-decreasing in x*

$$\gamma_i(\theta) \leq \gamma_i(\theta') \text{ for all } i \in N.$$

The following example will show that allocation rule *PUC* does not satisfy monotonicity to profit on Θ^L.

Example 6. *Consider an availability situation $\theta \in \Theta^L$ with $N = \{1, 2, 3\}$ and $A_1 = 0.6$, $A_2 = 0.7$ and $A_3 = 0.5$ and $p_1 = 1$, $p_2 = 3$ and $p_3 = 9$. Moreover, consider situation $\theta' \in \Theta^L$, which coincides with θ except that $p'_1 = 10$ now. In Table 4, the four allocations regarding those two situations θ and θ' are depicted for all three players. Note that all numbers are rounded to two decimals.*

Table 4: Allocations for availability game

	i	OP_i	PA_i	PUC_i	SV_i		i	OP_i	PA_i	PUC_i	SV_i
	1	0.94	4.07	0.95	1.69		1	9.40	6.89	9.44	8.83
θ	2	2.82	4.75	2.88	3.54	θ'	2	2.82	8.04	2.87	4.38
	3	8.46	3.39	8.39	6.98		3	8.46	5.74	8.37	7.46

Allocation rule PUC does not satisfy monotonicity to profit, since

$$PUC_3(\theta) > 8.38 > PUC_3(\theta'). \qquad \diamond$$

Theorem 6. *Allocation rules OP, PA and SV satisfy monotonicity to profit on Θ.*

Proof : See Appendix.

4.2.3 Situation Symmetry

Suppose that two players have the same profit function and availability. Then, those players both generate the same profit and both help other players (in terms of availability) in the same way. Hence, it is natural to assume that those players expect the same allocation. We will investigate whether the allocation rules will indeed allocate the same to both players, i.e., satisfy situation symmetry. For this we will introduce some new definitions.

Definition 7. *For any availability situation $\theta \in \Theta$, players $i, j \in N$ with $i \neq j$ are called situation symmetric if*

$$P_i(x) = P_j(x) \text{ for all } x \in [0, 1] \text{ and } A_i = A_j.$$

Definition 8. *An allocation rule γ satisfies situation symmetry on $D \subseteq \Theta$ if for all $\theta \in D$ and all situation symmetric players $i, j \in N$ with $i \neq j$ it holds that*

$$\gamma_i(\theta) = \gamma_j(\theta).$$

Theorem 7. *Allocation rules OP, PA, PUC and SV satisfy situation symmetry on Θ.*

Proof : See Appendix.

4.2.4 Game Symmetry

Suppose that player i and player j have the same individual profit, but not necessarily the same profit functions and availabilities. Moreover, assume that the total profit of any coalition including player i equals the total profit of the same coalition including player j rather than i. So, both players are symmetric, but now in terms of the corresponding availability game. Hence, it is natural to assume that both players expect the same allocation. We will investigate whether the allocation rules will allocate the same to both players, i.e., satisfy game symmetry. We will first introduce the definition of symmetric players in terms of availability games.

Definition 9. *For any availability situation $\theta \in \Theta$ players $i, j \in N$ with $i \neq j$ are called game symmetric if for the corresponding availability game (N, v^θ)*

$$v^\theta(S \cup \{i\}) = v^\theta(S \cup \{j\}) \ \ \forall S \subseteq N \backslash \{i, j\}.$$

Definition 10. *An allocation rule γ satisfies game symmetry on $D \subseteq \Theta$ if for all $\theta \in D$ and all game symmetric players $i, j \in N$ with $i \neq j$ it holds that*

$$\gamma_i(\theta) = \gamma_j(\theta).$$

The following example will show that allocation rules OP, PA and PUC do not satisfy game symmetry on Θ^L.

Example 7. *Consider a linear availability situation $\theta \in \Theta^L$ with $N = \{1,2,3\}$ and $A_1 = 0.7$, $A_2 = 0.8$ and $A_3 = 0.9$ $p_1 = 9, p_2 = 40$ and $p_3 = 7$. Then $v_1^\theta(\{1\}) = 6.3 = v_3^\theta(\{3\})$ and $v^\theta(\{1,2\}) = 49 \cdot 0.94 = 47 \cdot 0.98 = v^\theta(\{2,3\})$ and thus we can conclude that player 1 and 3 are symmetric. The corresponding allocations are presented in Table 5. All numbers are rounded to two decimals.*

Table 5: Allocations for availability game

	i	OP_i	PA_i	PUC_i	SV_i
	1	8.95	16.24	8.92	9.18
θ	2	39.76	18.55	39.76	37.30
	3	6.96	20.87	6.98	9.18

The allocation rules OP, PA and PUC do not satisfy game symmetry, since

$$OP_1(\theta) > 7.00 > OP_3(\theta),$$
$$PA_1(\theta) < 20.00 < PA_3(\theta),$$
$$PUC_1(\theta) > 7.00 > PUC_3(\theta).$$

◇

Theorem 8. *Allocation rule SV satisfies game symmetry on Θ.*

Proof : Let $\theta \in \Theta$ be an availability situation. Moreover, let $i, j \in N$ with $i \neq j$ be two game symmetric players in (N, v^θ). Following Shapley (1953), it holds that $\Phi_i(N, v^\theta) = \Phi_j(N, v^\theta)$. As a consequence, $SV_i(\theta) = \Phi(N, v^\theta) = \Phi_j(N, v^\theta) = SV_j(\theta)$, which concludes that SV satisfies game symmetry. □

4.2.5 The core

In Section 3.2 we already investigated the non-emptiness of the core. This result was based on finding a payoff vector that always belongs to the core. Now, we will investigate whether the payoff vectors resulting from the allocation rules are always members of the core as well.

The following example will show that there exists an availability situation $\theta \in \Theta$ for which payoff vectors $PA(\theta)$, $PUC(\theta)$ and $SV(\theta)$ are not core elements.

Example 8. *Consider the availability situation of Example 4. Then, the corresponding game* (N, v^θ) *is given by* $v^\theta(\{1\}) = v^\theta(\{2\}) = v^\theta(\{3\}) = \frac{1}{2}$, $v^\theta(\{1,3\}) = v^\theta(\{2,3\}) = 1\frac{1}{2}$, $v^\theta(\{1,2\}) = 1$ *and* $v^\theta(\{1,2,3\}) = 2$. *The payoff vectors resulting from allocation rules PA, PUC and SV (see Table 2) are not elements of the core, since*

$$PA_1(\theta) + PA_3(\theta) = \frac{2}{3} + \frac{2}{3} < 1\frac{1}{2} = v^\theta(\{1,3\}),$$
$$PUC_1(\theta) + PUC_3(\theta) = \frac{2}{3} + \frac{2}{3} < 1\frac{1}{2} = v^\theta(\{1,3\}),$$
$$SV_1(\theta) + SV_3(\theta) = \frac{7}{12} + \frac{10}{12} = 1\frac{5}{12} < 1\frac{1}{2} = v^\theta(\{1,3\}). \qquad \diamond$$

Theorem 9. *For every availability situation* $\theta \in \Theta$ *it holds that*

$$OP(\theta) \in C(N, v^\theta).$$

Proof : See proof of Theorem 1 where $OP_i(\theta) = x_i$ for all $i \in N$. □

The following example will show that there exists a linear availability situation $\theta \in \Theta^L$ for which payoff vector $PA(\theta)$ and $SV(\theta)$ are not core elements.

Example 9. *Consider the (linear) availability situation* $\theta \in \Theta^L$ *of Example 1. The related allocations are presented in Table 6. All values are rounded to two decimal places.*

Table 6: Allocations for availability game

	i	OP_i	PA_i	PUC_i	SV_i
	1	0.98	2.94	0.98	1.28
θ	2	1.96	4.41	1.99	2.95
	3	6.86	2.45	6.83	5.57

The payoff vectors resulting from allocation rules PA and SV are not elements of the core, since

$$PA_2(\theta) + PA_3(\theta) < 4.42 + 2.46 = 6.88 < v^\theta(\{23\}),$$
$$SV_2(\theta) + SV_3(\theta) < 2.96 + 5.58 = 8.54 < v^\theta(\{23\}). \qquad \diamond$$

For the upcoming theorem, the following lemma will be used.

Lemma 3. *Let $\theta \in \Theta^L$ be a linear availability situation with $x_i = 1 - A_i$ for all $i \in N$. Then for all $S \subseteq N$ it holds that*

$$\sum_{i \in S} p_i \left(\prod_{j \in S} x_j \right) \geq \frac{\sum_{i \in S} p_i x_i}{\sum_{j \in N} p_j x_j} \sum_{i \in N} p_i \left(\prod_{j \in N} x_j \right).$$

Now, it is possible to show that for every linear availability situation $\theta \in \Theta^L$ payoff vectors $PUC(\theta)$ and $OP(\theta)$ are core members.

Theorem 10. *For every availability situation $\theta \in \Theta^L$ it holds that*

$$PUC(\theta), OP(\theta) \in C(N, v^\theta).$$

Proof : See Appendix.

Following Shapley (1953), the Shapley value is a member of the core if the corresponding game is convex. In Theorem 3 necessary and sufficient conditions are given for convexity of games associated with linear availability situations.

Corollary 5. *For linear availability situations $\theta \in \Theta^L$ with $\mathcal{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ and $i \neq j$, $SV(\theta)$ is a member of the core of (N, v^θ).*

5 Conclusions

In this paper, an environment was considered in which several independent service providers can collaborate by pooling their critical, low-utilization resources that are subject to unavailability. We examined the allocation of the collective cost savings for such pooled situation by studying an associated cooperative game. For this game, we proved non-emptiness of the core, presented a population monotonic allocation scheme, and showed convexity under some conditions. Moreover, we discussed four allocation rules and investigated whether they satisfy intuitive properties as monotonicity to availability, monotonicity to profit, situation symmetry and game symmetry. Next to that, we investigated whether the payoff vectors resulting for those allocation rules are core members. In Table 7 and Table 8 all results are summarized together. It turns out that none of the four allocation rules satisfies all properties. However, in terms of the underlying properties, allocation rule Own Profit (*OP*) is preferable to allocation rule Proportional to Unavailability costs (*PUC*) and to allocation rule Proportional to Availability (*PA*). Finally, Allocation rule Shapley Value (*SV*) is preferable to *PA*.

Table 7: Results for availability situations

Properties	*OP*	*PA*	*PUC*	*SV*
Monotonicity to availability	✓	×	×	×
Monotonicity to profit	✓	✓	×	✓
Situation symmetry	✓	✓	✓	✓
Game symmetry	×	×	×	✓
Member of the core	✓	×	×	×

✓: satisfies property

× : does not (always) satisfy property

Table 8: Results for lineair availability situations

Properties	*OP*	*PA*	*PUC*	*SV*
Monotonicity to availability	✓	×	✓	×
Monotonicity to profit	✓	✓	×	✓
Situation symmetry	✓	✓	✓	✓
Game symmetry	×	×	×	✓
Member of the core	✓	×	✓	$\times^{\star}$

$\star$: satisfies property if conditions of Corollary 5 hold.

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6 Appendix

Proof of Theorem 3

Proof : Let $\theta \in \Theta^L$ be a linear availability situation with $|N| \geq 2$ and (N, v^θ) be the corresponding availability game. We will show that the corresponding availability game is convex if and only if $\mathcal{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$.

($\Rightarrow$) Suppose the availability game is convex, i.e. (Shapley (1953)),

$$v^\theta(S \cup \{i,j\}) - v^\theta(S \cup \{j\}) - (v^\theta(S \cup \{i\}) - v^\theta(S)) \geq 0 \tag{4}$$

for all $i, j \in N$ with $i \neq j$ and all $S \subseteq N \backslash \{i,j\}$. Let $i, j \in N$ with $i \neq j$ and $S \subseteq N \backslash \{i,j\}$. Based on (4) it holds that

$$\begin{aligned}
0 &\leq v^\theta(S \cup \{i,j\}) - v^\theta(S \cup \{j\}) - (v^\theta(S \cup \{i\}) - v^\theta(S)) \\
&= \sum_{k \in S \cup \{i,j\}} p_k \left(1 - \prod_{l \in S \cup \{i,j\}} (1 - A_l)\right) - \sum_{k \in S \cup \{j\}} p_k \left(1 - \prod_{l \in S \cup \{j\}} (1 - A_l)\right) \\
&\quad - \sum_{k \in S \cup \{i\}} p_k \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) + \sum_{k \in S} p_k \left(1 - \prod_{l \in S} (1 - A_l)\right) \\
&= \sum_{k \in S \cup \{j\}} p_k \left(A_i \prod_{l \in S \cup \{j\}} (1 - A_l)\right) + p_i \left(1 - \prod_{l \in S \cup \{i,j\}} (1 - A_l)\right) \\
&\quad - \sum_{k \in S} p_k \left(A_i \prod_{l \in S} (1 - A_l)\right) - p_i \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) \\
&= \prod_{l \in S \cup \{j\}} (1 - A_l) \left(\sum_{k \in S \cup \{j\}} p_k A_i - p_i (1 - A_i)\right) \\
&\quad - \prod_{l \in S} (1 - A_l) \left(\sum_{k \in S} p_k A_i - p_i (1 - A_i)\right) \\
&= \prod_{l \in S} (1 - A_l) \left((1 - A_j) \left(\sum_{k \in S \cup \{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k \in S \cup \{i\}} p_k A_i - p_i\right)\right),
\end{aligned}$$

where the first equality follows by definition. The second equality follows by combining all terms $k \in S \cup \{j\}$ from the first and second summation into one summation and combining all terms $k \in S$ from the third and fourth summation into one summation. In the two new summations we combine the product terms and use that $A_i = 1 - (1 - A_i)$. Finally, we write down the terms that are left from the original summations. In the third equality the product term $\prod_{l \in S \cup j}(1 - A_l)$ is taken out of the first and second term and the product term $\prod_{l \in S}(1 - A_l)$ is taken out of the third and fourth term. Moreover, $p_i \cdot 1$ and $-p_i \cdot 1$ cancel out against each other. In the fourth equality the product term $\prod_{l \in S}(1 - A_l)$ is taken out of the whole equality and $-p_i(1 - A_i)$ is written as $p_i A_i - p_i$, where $p_i A_i$ is finally included into the summation.

As $A_i \in [0, 1)$ for all $i \in N$, it holds that $\prod_{l \in S}(1 - A_l) > 0$. If the last expression is divided by $\prod_{l \in S}(1 - A_l)$, we obtain

$$
\begin{aligned}
0 &\leq (1 - A_j)\left(\sum_{k \in S \cup \{i,j\}} p_k A_i - p_i\right) - \left(\sum_{k \in S \cup \{i\}} p_k A_i - p_i\right) \\
&= p_j A_i - A_j\left(\sum_{k \in S \cup \{i,j\}} p_k A_i - p_i\right).
\end{aligned}
$$

This is equivalent to

$$
A_j\left(\sum_{k \in S \cup \{i,j\}} p_k A_i - p_i\right) - p_j A_i \leq 0. \tag{5}
$$

As $i, j \in N$ with $i \neq j$ and $S \subseteq N \backslash \{i, j\}$ were chosen arbitrarily, (5) holds for any $i, j \in N$ with $i \neq j$ and all $S \subseteq N \backslash \{i, j\}$. In particular, (5) holds for any $i, j \in N$ with $i \neq j$ and $S = N \backslash \{i, j\}$. For $S = N \backslash \{i, j\}$ the left side of (5) coincides with $\mathscr{L}_{ij}(\theta)$ and thus $\mathscr{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$.

($\Leftarrow$) Now, we assume that

$$
\mathcal{L}_{ij}(\theta) \leq 0
$$

for all $i, j \in N$ with $i \neq j$. Then, for a given $i, j \in N$ with $i \neq j$ it holds that

$$
A_j\left(\sum_{k \in N} p_k A_i - p_i\right) - p_j A_i \leq 0.
$$

Now, let $S \subseteq N \backslash \{i, j\}$. As $\sum_{k \in S \cup \{i,j\}} p_k A_i \leq \sum_{k \in N} p_k A_i$, we can conclude that

$$
A_j\left(\sum_{k \in S \cup \{i,j\}} p_k A_i - p_i\right) - p_j A_i \leq A_j\left(\sum_{k \in N} p_k A_i - p_i\right) - p_j A_i \leq 0.
$$

This implies that

$$\begin{aligned}
0 &\leq -A_j\left(\sum_{k\in S\cup\{i,j\}} p_kA_i - p_i\right) + p_jA_i \\
&= -A_j\left(\sum_{k\in S\cup\{i,j\}} p_kA_i - p_i\right) + \left(\sum_{k\in S\cup\{i,j\}} p_kA_i - p_i\right) \\
&- \left(\sum_{k\in S\cup\{i\}} p_kA_i - p_i\right) \\
&= (1-A_i)\left(\sum_{k\in S\cup\{i,j\}} p_kA_i - p_i\right) - \left(\sum_{k\in S\cup\{i\}} p_kA_i - p_i\right)
\end{aligned}$$

Multiplying the last expression by $\prod_{l\in S}(1-A_l) > 0$ results into

$$\prod_{l\in S}(1-A_l)\left((1-A_i)\left(\sum_{k\in S\cup\{i,j\}} p_kA_i - p_i\right) - \left(\sum_{k\in S\cup\{i\}} p_kA_i - p_i\right)\right) \geq 0.$$

From proof (⇒) we know that this inequality coincides with

$$v(S\cup\{i,j\}) - v(S\cup\{j\}) - (v(S\cup\{i\}) - v(S)) \geq 0 \tag{6}$$

As $i,j \in N$ with $i \neq j$ and $S \subseteq N\backslash\{i,j\}$ were chosen arbitrarily, we can conclude that (6) holds for any $i,j \in N$ with $i \neq j$ and all $S \subseteq N\backslash\{i,j\}$. This coincides with convexity, which concludes the proof. □

Proof of Corollary 2

Proof : Let $\theta \in \Theta^L$ be a lineair availability situation with $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$ and $1 > A_1 \geq A_2 \geq \ldots \geq A_n$. Let (N, v^θ) be the corresponding availability game. We will show that the corresponding availability game is convex if and only if $|N|A_1A_2 - A_1 - A_2 \leq 0$.

(⇒) Suppose the corresponding availability game is convex. Then, by Theorem 3, $\mathcal{L}_{ij}(\theta) \leq 0$ for all $i,j \in N$ with $i \neq j$ and so

$$\mathscr{L}_{12}(\theta) = A_2\left(|N|\overline{p}A_1 - \overline{p}\right) - \overline{p}A_1 \leq 0.$$

As $\overline{p} \in \mathbb{R}_+$, we derive

$$|N|A_1A_2 - A_1 - A_2 \leq 0,$$

which concludes this part of the proof.

($\Leftarrow$) Suppose that $|N|A_1A_2 - A_1 - A_2 \leq 0$. Then, it holds

$$A_1\left(\frac{1}{2}|N|A_2 - 1\right) + A_2\left(\frac{1}{2}|N|A_1 - 1\right) \leq 0. \tag{7}$$

As $0 \leq A_2 \leq A_1 < 1$, this implies that $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \leq 0$ or $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \geq 0$. We will now investigate those different cases.

Case 1 $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \leq 0$

As $\frac{1}{2}|N|A_1 - 1 \leq 0$, it holds that $\frac{1}{2}|N|A_j - 1 \leq \frac{1}{2}|N|A_1 - 1 \leq 0$ for all $j \in N$. As $A_i \in [0, 1)$ for all $i \in N$, it holds that $A_i(\frac{1}{2}|N|A_j - 1) \leq 0$ for all $i, j \in N$. Thus

$$A_i\left(\frac{1}{2}|N|A_j - 1\right) + A_j\left(\frac{1}{2}|N|A_i - 1\right) \leq 0.$$

Case 2 $\frac{1}{2}|N|A_2 - 1 \leq 0$ and $\frac{1}{2}|N|A_1 - 1 \geq 0$

(i) As $A_1(\frac{1}{2}|N|A_j - 1) \leq A_1(\frac{1}{2}|N|A_2 - 1)$ for all $j \in N\backslash 1$ and $A_j(\frac{1}{2}|N|A_1 - 1) \leq A_2(\frac{1}{2}|N|A_1 - 1)$ for all $j \in N\backslash 1$, it holds that

$$A_1(\frac{1}{2}|N|A_j - 1) + A_j(\frac{1}{2}|N|A_1 - 1) \leq A_1(\frac{1}{2}|N|A_2 - 1) + A_2(\frac{1}{2}|N|A_1 - 1) \leq 0$$

for all $j \in N\backslash 1$.

(ii) For $i \in N\backslash 1$ and $j \in N : j > i$, it holds that $\frac{1}{2}|N|A_j - 1 \leq \frac{1}{2}|N|A_i - 1 \leq \frac{1}{2}|N|A_2 - 1 \leq 0$. Thus

$$A_i\left(\frac{1}{2}|N|A_j - 1\right) + A_j\left(\frac{1}{2}|N|A_i - 1\right) \leq 0, \tag{8}$$

Combining Case 1 and Case 2, we conclude that (8) holds for all $i \in N$ and all $j \in N$ with $j > i$. Since $\mathscr{L}_{ij}(\theta) = \mathscr{L}_{ji}(\theta)$ for all $i, j \in N$ with $i \neq j$, (8) holds for all $i, j \in N$ with $i \neq j$. As multiplying (8) with $\overline{p} \in \mathbb{R}_+$ will not affect the right hand side of the inequality, it holds that $\mathscr{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$. By Theorem 3, the corresponding availability game is convex. $\square$

Proof of Corollary 3

Proof : Let $\theta \in \Theta^L$ be a lineair availability situation with $A_i = \overline{A} \in [0,1)$ for all $i \in N$ and $p_1 \leq p_2 \ldots \leq p_n$. Let (N, v^θ) be the corresponding availability game. We will show that the corresponding availability game is convex if and only if $\overline{A} \leq \frac{p_1+p_2}{\sum_{i\in N} p_i}$.

($\Rightarrow$) Suppose the corresponding availability game is convex. Then, by Theorem 3, $\mathcal{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$ and so

$$\mathscr{L}_{12}(\theta) = \overline{A}\left(\sum_{i\in N} p_i\overline{A} - p_1\right) - p_2\overline{A} \leq 0.$$

After some rewriting, we derive

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{i\in N} p_i},$$

which concludes this part of the proof.

($\Leftarrow$) Suppose that $0 \leq \overline{A} \leq \frac{p_1+p_2}{\sum_{i\in N} p_i}$. After some rewriting, we derive

$$\overline{A}\left(\sum_{i\in N} p_i\overline{A} - p_1\right) - p_2\overline{A} \leq 0. \tag{9}$$

The left hand side of (9) coincides with $\mathscr{L}_{12}(\theta)$, and so $\mathscr{L}_{12}(\theta) \leq 0$. Now, observe that

$$\begin{aligned} 0 \geq \mathcal{L}_{12}(\theta) = \overline{A}\left(\sum_{k\in N} p_k\overline{A} - p_1\right) - p_2\overline{A} &= \overline{A}^2 \sum_{k\in N} p_k - \overline{A}(p_1 + p_2) \\ &\geq \overline{A}^2 \sum_{k\in N} p_k - \overline{A}(p_i + p_j) \\ &= \mathscr{L}_{ij}(\theta) \end{aligned}$$

for all $i, j \in N$ with $i \neq j$. This implies that $\mathscr{L}_{ij}(\theta) \leq 0$ for all $i, j \in N$ with $i \neq j$. By Theorem 3, the corresponding availability game is convex. $\square$

Proof of Corollary 4

Proof : Let $\theta \in \Theta^L$ be a lineair availability situation with $A_i = \overline{A} \in [0,1)$ for all $i \in N$ and $p_i = \overline{p} \in \mathbb{R}_+$ for all $i \in N$. Let (N, v^θ) be the corresponding availability game. We will show that the corresponding availability game is convex if and only if $\overline{A} \leq \frac{2}{|N|}$.

($\Rightarrow$) Suppose the corresponding availability game is convex. Then, by Corollary 3, it holds that

$$\overline{A} \leq \frac{p_1 + p_2}{\sum_{k \in N} p_k} = \frac{2\overline{p}}{|N|\overline{p}} = \frac{2}{|N|},$$

which concludes the first part of the proof.

($\Leftarrow$) Suppose that $\overline{A} \leq \frac{2}{|N|}$. This implies that

$$\overline{A} \leq \frac{2}{|N|} = \frac{2\overline{p}}{|N|\overline{p}} = \frac{p_1 + p_2}{\sum_{k \in N} p_k},$$

and thus, by Corollary 3, the corresponding game is convex. □

Proof of Theorem 4

Proof : Let $\theta \in \Theta$ be an availability situation and $\theta' \in \Theta$ be another availability situation that coincides with θ except for the availability of player j, i.e., $A_j \leq A'_j$. Then, it holds for any player $i \in N$ that

$$\begin{aligned}
OP_i(\theta) &= P_i \left(1 - \prod_{k \in N} (1 - A_k)\right) \\
&= P_i \left(1 - \prod_{k \in N \setminus \{j\}} (1 - A_k)(1 - A_j)\right) \\
&= P'_i \left(1 - \prod_{k \in N \setminus \{j\}} (1 - A'_k)(1 - A_j)\right) \\
&\leq P'_i \left(1 - \prod_{k \in N \setminus \{j\}} (1 - A'_k)(1 - A'_j)\right) \\
&= OP_i(\theta').
\end{aligned}$$

where the third equality results from $A_k = A'_k$ for all $k \neq j$ and $P_i = P'_i$ for all $i \in N$. The inequality results from $0 \leq A_j \leq A'_j \leq 1$ with $0 \leq \prod_{k \in N \setminus \{j\}} (1 - A'_k) \leq 1$, and the fact that P_i is non-decreasing. $\square$

Proof of Theorem 5

Proof : *(i) OP*. From Theorem 4 it follows that allocation rule OP satisfies monotonicity to availability on Θ. As $\Theta^L \subseteq \Theta$, allocation rule OP satisfies monotonicity to availability on Θ^L as well.

(ii) PUC. Let $\theta \in \Theta^L$ be a linear availability situation and $\theta' \in \Theta^L$ be another linear availability situation that only deviates in the availability of player $j \in N$ with $A_j \leq A'_j$. We claim that the derivative of $PUC_i(\theta)$ for any player $i \in N$ is non-negative with respect to availability A_j. Note that $P_i(1) - P_i(A_i) = p_i - p_i A_i = p_i(1 - A_i)$ for all $i \in N$.

Allocation $PUC_i(\theta)$ for player $i \in N$ can be rewritten as

$$\begin{aligned}
PUC_i(\theta) &= p_i A_i + \frac{p_i(1 - A_i)}{\sum_{k \in N} p_k(1 - A_k)} \left(\sum_{t \in N} p_t \left(1 - \prod_{k \in N} (1 - A_k) \right) - \sum_{l \in N} p_l A_l \right) \\
&= p_i A_i + \left(1 - \frac{\sum_{l \in N \setminus \{i\}} p_l (1 - A_l)}{\sum_{l \in N} p_l (1 - A_l)} \right) \times \\
&\qquad \left(\sum_{t \in N} p_t \left(1 - A_t - \prod_{k \in N} (1 - A_k) \right) \right) \\
&= p_i A_i + \sum_{t \in N} p_t \left(1 - A_t - \prod_{k \in N} (1 - A_k) \right) - \sum_{l \in N \setminus \{i\}} p_l (1 - A_l) \\
&\qquad + \frac{\sum_{l \in N \setminus \{i\}} p_l (1 - A_l)}{\sum_{l \in N} p_l (1 - A_l)} \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \\
&= p_i A_i + p_i (1 - A_i) - \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \\
&\qquad + \frac{\sum_{l \in N \setminus \{i\}} p_l (1 - A_l)}{\sum_{l \in N} p_l (1 - A_l)} \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \\
&= p_i - \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \left(1 - \frac{\sum_{l \in N \setminus \{i\}} p_l (1 - A_l)}{\sum_{l \in N} p_l (1 - A_l)} \right)
\end{aligned}$$

$$= p_i - \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \left(\frac{p_i(1 - A_i)}{\sum_{l \in N} p_l(1 - A_l)} \right).$$

As at least for one player $k \in N$, $A_k < 1$, function $PUC_i(\theta)$ is continuous and differentiable in A_j. For player j, the derivative of $PUC_j(\theta)$ to A_j is given by

$$\begin{aligned}
&\frac{d}{dA_j} PUC_j(\theta) \\
&\quad = -\frac{\sum_{l \in N} p_l(1 - A_l) \cdot (-p_j) - p_j(1 - A_j) \cdot (-p_j)}{\left(\sum_{l \in N} p_l(1 - A_l)\right)^2} \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \\
&\qquad - \frac{p_j(1 - A_j)}{\sum_{l \in N} p_l(1 - A_l)} \sum_{t \in N} p_t \prod_{k \in N \setminus \{j\}} (1 - A_k) \cdot (-1) \\
&\quad = \frac{p_j}{\left(\sum_{l \in N} p_l(1 - A_l)\right)^2} \left(\sum_{l \in N \setminus \{j\}} p_l(1 - A_l) \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \right. \\
&\qquad \left. + \sum_{l \in N} p_l(1 - A_l) \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \right) \\
&\quad = \frac{p_j \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k)}{\left(\sum_{l \in N} p_l(1 - A_l)\right)^2} \left(\sum_{l \in N \setminus \{j\}} p_l(1 - A_l) + \sum_{l \in N} p_l(1 - A_l) \right) \geq 0.
\end{aligned}$$

Note that all terms are non-negative and thus the derivative is non-negative as well. Hence, $PUC_j(\theta)$ is non-decreasing in A_j. This implies that $PUC_j(\theta) \leq PUC_j(\theta')$. Taking the derivative of $PUC_i(\theta)$ to A_j with $i \in N \setminus \{j\}$ gives

$$\begin{aligned}
\frac{d}{dA_j} PUC_i(\theta) &= -\frac{0 - (p_i(1 - A_i) \cdot (-p_j))}{\left(\sum_{l \in N} p_l(1 - A_l)\right)^2} \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \\
&\quad - \frac{p_i(1 - A_i)}{\sum_{l \in N} p_l(1 - A_l)} \sum_{t \in N} p_t \prod_{k \in N \setminus \{j\}} (1 - A_k) \cdot (-1) \\
&= \frac{p_i(1 - A_i)}{\left(\sum_{l \in N} p_l(1 - A_l)\right)^2} \left(-p_j \sum_{t \in N} p_t \prod_{k \in N} (1 - A_k) \right. \\
&\qquad \left. + \sum_{t \in N} p_t \prod_{k \in N \setminus \{j\}} (1 - A_k) \sum_{l \in N} p_l(1 - A_l) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p_i(1-A_i)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(\sum_{t\in N} p_t \prod_{k\in N\setminus\{j\}}(1-A_k)\right) \\
&\qquad \times\left(-p_j(1-A_j)+\sum_{l\in N} p_l(1-A_l)\right) \\
&= \frac{p_i(1-A_i)}{\left(\sum_{l\in N} p_l(1-A_l)\right)^2}\left(\sum_{l\in N} p_l \prod_{k\in N\setminus\{j\}}(1-A_k)\right) \\
&\qquad \times\left(\sum_{l\in N\setminus\{j\}} p_l(1-A_l)\right) \geq 0.
\end{aligned}
$$

Note that all terms are non-negative and thus the derivative is non-negative as well. Hence, $PUC_i(\theta)$ is non-decreasing in A_j for all $i \in N\setminus\{j\}$. We conclude that $PUC_i(\theta) \leq PUC_i(\theta')$ for all $i \in N$. □

Proof of Theorem 6

Proof : Let $\theta, \theta' \in \Theta$ be two availability situations where θ and θ' coincide except for the profit of player j with $P'_j(x) - P_j(x)$ non-decreasing in x. As $P'_k(x) - P_k(x) = 0$ for all $k \in N\setminus j$, it holds that $P'_i(x) \geq P_i(x)$ for all $i \in N$. Hence, it holds for all $i \in N$ that

$$
\begin{aligned}
OP_i(\theta) &= P_i\left(1-\prod_{k\in N}(1-A_k)\right) \\
&\leq P'_i\left(1-\prod_{k\in N}(1-A'_k)\right) = OP_i(\theta')
\end{aligned}
$$

given that $A_k = A'_k$ for all $k \in N$.

In the same line, it holds that

$$
\begin{aligned}
PA_i(\theta) &= \frac{A_i}{\sum_{k\in N} A_k}\sum_{k\in N} P_k\left(1-\prod_{h\in N}(1-A_h)\right) \\
&\leq \frac{A'_i}{\sum_{k\in N} A'_k}\sum_{k\in N} P'_k\left(1-\prod_{h\in N}(1-A'_h)\right) = PA_i(\theta'),
\end{aligned}
$$

given that $A_k = A'_k$ for all $k \in N$.

Finally, let $i \in N$ and $S \subseteq N \backslash \{i\}$. Then

$$v^{\theta'}(S \cup \{i\}) - v^{\theta'}(S) = \sum_{k \in S \cup \{i\}} P'_k \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) - \sum_{k \in S} P'_k \left(1 - \prod_{l \in S} (1 - A_l)\right)$$

$$\begin{aligned}
&= \sum_{k \in S} \left(P'_k \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) - P'_k \left(1 - \prod_{l \in S} (1 - A_l)\right)\right) + P'_i \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) \\
&\geq \sum_{k \in S} \left(P_k \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) - P_k \left(1 - \prod_{l \in S} (1 - A_l)\right)\right) + P_i \left(1 - \prod_{l \in S \cup \{i\}} (1 - A_l)\right) \\
&= v^{\theta}(S \cup \{i\}) - v^{\theta}(S).
\end{aligned}$$

where the inequality holds, as

i) if $j \in S$ (and thus $j \neq i$) then $P'_j(y) - P'_j(x) \geq P_j(y) - P_j(x)$ for $y \geq x$ and $P'_l = P_l$ for all $l \in S \backslash \{j\}$ and $P'_i = P_i$.

ii) if $j \notin S$ and $i = j$ then $P'_l = P_l$ for all $l \in S$ and $P'_i \geq P_i$.

iii) if $j \notin S$ and $i \neq j$ then $P'_l = P_l$ for all $l \in S$ and $P'_i = P_i$ and so the inequality becomes equality.

As $v^{\theta'}(S \cup \{i\}) - v^{\theta'}(S) \geq v^{\theta}(S \cup \{i\}) - v^{\theta}(S)$ for any $i \in N$ and $S \subseteq N \backslash \{i\}$ and given that the Shapley value of a cooperative game is a weighted average over those marginal contributions, it follows that

$$SV_i(\theta) = \Phi_i(N, v^{\theta}) \leq \Phi(N, v^{\theta'}) = SV_i(\theta') \text{ for all } i \in N,$$

which concludes the proof. □

Proof of Theorem 7

Proof : Let $\theta \in \Theta$ be an availability situation and $i, j \in N$ with $i \neq j$ two players that are situation symmetric. For allocation rule OP it holds that

$$OP_i(\theta) = P_i\left(1 - \prod_{k \in N}(1 - A_i)\right) = P_j\left(1 - \prod_{k \in N}(1 - A_i)\right) = OP_j(\theta),$$

as $P_i(x) = P_j(x)$ for all $x \in [0,1]$. For allocation rule PA, it holds that

$$PA_i(\theta) = \frac{A_i}{\sum_{k \in N} A_k} v^\theta(N) = \frac{A_j}{\sum_{k \in N} A_k} v^\theta(N) = PA_j(\theta),$$

as $A_i = A_j$. For allocation rule PUC, it holds that

$$\begin{aligned}
PUC_i(\theta) &= v^\theta(\{i\}) + \frac{P_i(1) - P_i(A_i)}{\sum_{k \in N} P_k(1) - P_k(A_k)}\left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\})\right) \\
&= P_i(A_i) + \frac{P_i(1) - P_i(A_i)}{\sum_{k \in N} P_k(1) - P_k(A_k)}\left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\})\right) \\
&= P_j(A_j) + \frac{P_j(1) - P_j(A_j)}{\sum_{k \in N} P_k(1) - P_k(A_k)}\left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\})\right) \\
&= v^\theta(\{j\}) + \frac{P_j(1) - P_j(A_j)}{\sum_{k \in N} P_k(1) - P_k(A_k)}\left(v^\theta(N) - \sum_{l \in N} v^\theta(\{l\})\right) \\
&= PUC_j(\theta),
\end{aligned}$$

As $P_i(x) = P_j(x)$ for all $x \in [0,1]$ and $A_i = A_j$. Finally, for allocation rule SV, it holds that $P_i(x) = P_j(x)$ for all $x \in [0,1]$ and $A_i = A_j$. This implies that $v^\theta(S \cup \{i\}) = v^\theta(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Based on Definition 9, player i and j are game symmetric. Based on Theorem 8[2], we can conclude that $SV_i(\theta) = SV_j(\theta)$. This concludes the proof. □

Proof of Lemma 3

Proof : Let $\theta \in \Theta^L$ be a linear availability situation and $x_i = 1 - A_i$ for all $i \in N$. Then, it holds that

$$\sum_{i \in S} p_i x_i \sum_{j \in S} p_j \left(1 - \prod_{k \in N \backslash S} x_k\right) \geq 0.$$

[2]This theorem will be proven later on.

Moreover, it holds that

$$\sum_{i\in N\setminus S} p_i x_i \sum_{k\in S} p_k - \sum_{i\in N\setminus S} p_i \prod_{j\in N\setminus S} x_j \sum_{k\in S} p_k x_k \geq 0.$$

Now, when both parts are summed, we obtain

$$\begin{aligned}
0 \leq \sum_{i\in S} p_i x_i \sum_{j\in S} p_j \left(1 - \prod_{k\in N\setminus S} x_k\right) &+ \sum_{i\in N\setminus S} p_i x_i \sum_{k\in S} p_k \\
&- \sum_{i\in N\setminus S} p_i \prod_{j\in N\setminus S} x_j \sum_{k\in S} p_k x_k \\
= \sum_{i\in S} p_i x_i \sum_{j\in S} p_j - \sum_{i\in S} p_i x_i \sum_{j\in S} p_j \prod_{k\in N\setminus S} x_k &+ \sum_{i\in N\setminus S} p_i x_i \sum_{k\in S} p_k \\
&- \sum_{i\in N\setminus S} p_i \prod_{j\in N\setminus S} x_j \sum_{k\in S} p_k x_k
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i\in N} p_i x_i \sum_{j\in S} p_j - \sum_{i\in S} p_i x_i \sum_{j\in S} p_j \prod_{k\in N\setminus S} x_k - \sum_{i\in N\setminus S} p_i \prod_{j\in N\setminus S} x_j \sum_{k\in S} p_k x_k \\
&= \sum_{i\in N} p_i x_i \sum_{j\in S} p_j - \sum_{i\in S} p_i x_i \sum_{j\in S} p_j \prod_{k\in N\setminus S} x_k - \sum_{k\in S} p_k x_k \sum_{i\in N\setminus S} p_i \left(\prod_{j\in N\setminus S} x_j\right) \\
&= \sum_{i\in N} p_i x_i \sum_{j\in S} p_j - \sum_{i\in S} p_i x_i \sum_{j\in N} p_j \left(\prod_{k\in N\setminus S} x_k\right),
\end{aligned}$$

where the equalities hold by rewriting. From the last expression, we derive

$$\sum_{i\in N} p_i x_i \sum_{j\in S} p_j \geq \sum_{i\in S} p_i x_i \sum_{j\in N} p_j \left(\prod_{k\in N\setminus S} x_k\right).$$

Multiplying both sides by $\prod_{j\in S} x_j \geq 0$ and subsequently dividing both sides by $\sum_{j\in N} p_j x_j$ gives

$$\sum_{i\in S} p_i \prod_{j\in S} x_i \geq \frac{\sum_{i\in S} p_i x_i}{\sum_{j\in N} p_j x_j} \sum_{i\in N} p_i \prod_{j\in N} x_i,$$

which concludes the proof. □

Proof of Theorem 10

Proof **:** Let $\theta \in \Theta^L$ be a linear availability situation. Note that $P_i(1) - P_i(A_i) = p_i - p_i A_i = p_i(1 - A_i)$ for all $i \in N$.

(i) PUC. It holds that

$$\begin{aligned}\sum_{i\in N} PUC_i &= \sum_{i\in N}\left(v^\theta(\{i\}) + \frac{p_i(1-A_i)}{\sum_{j\in N} p_j(1-A_j)}\left(v^\theta(N) - \sum_{k\in N} v^\theta(\{k\})\right)\right)\\ &= \sum_{i\in N} v^\theta(\{i\}) + v^\theta(N) - \sum_{k\in N} v^\theta(\{k\})\\ &= v^\theta(N),\end{aligned}$$

and thus $PUC(\theta)$ is efficient. Secondly, let $M \subseteq N$, then

$$\begin{aligned}v^\theta(M) &= \sum_{i\in M} p_i\left(1 - \prod_{j\in M}(1-A_j)\right)\\ &= \sum_{i\in M} p_i A_i + \sum_{i\in M} p_i(1-A_i) - \sum_{i\in M} p_i \prod_{j\in M}(1-A_j)\\ &\leq \sum_{i\in M} v^\theta(\{i\}) + \sum_{i\in M} p_i(1-A_i) - \frac{\sum_{i\in M} p_i(1-A_i)}{\sum_{k\in N} p_k(1-A_k)} \sum_{i\in N} p_i \prod_{j\in N}(1-A_j)\\ &= \sum_{i\in M} v^\theta(\{i\}) + \frac{\sum_{i\in M} p_i(1-A_i)}{\sum_{k\in N} p_k(1-A_k)}\\ &\qquad\qquad \times\left(\sum_{k\in N} p_k(1-A_k) - \sum_{i\in N} p_i \prod_{j\in N}(1-A_j)\right)\\ &= \sum_{i\in M} v^\theta(\{i\}) + \frac{\sum_{i\in M} p_i(1-A_i)}{\sum_{k\in N} p_k(1-A_k)}\\ &\qquad\qquad \times\left(\sum_{k\in N} p_k(1 - \prod_{j\in N}(1-A_j)) - \sum_{k\in N} p_k A_k\right)\\ &= \sum_{i\in M}\left(v^\theta(\{i\}) + \frac{p_i(1-A_i)}{\sum_{k\in N} p_k(1-A_k)}\left(v^\theta(N) - \sum_{k\in N} v^\theta(\{k\})\right)\right)\\ &= \sum_{i\in M} PUC_i,\end{aligned}$$

where the inequality is a result of Lemma 3 with $S = M$ and $x_j = 1 - A_j$ for all $j \in N$. Hence, $PUC(\theta)$ is also stable and thus a member of the core.

(ii) OP. From Theorem 9 it follows that $OP(\theta) \in C(N, v^\theta)$ for every $\theta \in \Theta$. Note that $\theta \in \Theta^L \subseteq \Theta$. Hence, $OP(\theta) \in C(N, v^\theta)$ for all $\theta \in \Theta^L$. □