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Abstract

Uptime-guarantee maintenance contracts reflect the obligation of service providers to maintain full functionality of equipment in certain fractions of working time in a period. In this paper, we study the optimal design of uptime-guarantee contracts for a service provider who does not know the customer's exact valuation for having the equipment up and running per unit of working time. We first consider the case of service provider offering a single contract and obtain the optimal contract for situations where the customer's valuation has Increasing Generalized Failure Rate (IGFR) distribution functions. In the second part of the paper we consider the case where service provider proposes a menu of contracts with various guaranteed uptime levels. Under certain conditions on costs of providing different contracts, we find the optimal prices for uptime-guarantee contract menus. We provide necessary and sufficient conditions for the existence of optimal contract menus.

1 Introduction

In many industries, steady execution of value added activities depends on uninterrupted performance of sophisticated equipment whose maintenance must be outsourced. Imperfect reliability of critical equipment threats the revenue stream of their owners. To hedge the risk of break downs, owners would be willing to pay premiums for services which increase the reliability of their equipment. On the other hand, faced by saturated market and fierce competition, many traditional core manufacturing companies—e.g., General Electric Co., IBM Corp., Siemens AG and Hewlett-Packard Co.—are reinventing their services as key sources of revenue (Sawhney et al., 2003). When managed correctly, maintenance contracts can leverage the customers' requirements for more reliable equipment and the service providers' revenue diversification strategies.

A standard quality control indicator which may be used to determine the effectiveness of a maintenance service contract is the equipment's *uptime*, that is, the percentage of time that the device will be operational and ready to use (Chan, 2003). Accordingly, maintenance contracts could be designed to specifically guarantee uptime of equipment. In this sense, *uptime-guarantee* contracts are special form of performance-based contracts (see Selviaridis and Wynstra (2015) for a review). An uptime-guarantee contract specifies the service provider's commitment to ensure that the equipment under contract is functioning properly at least in a certain percentage of working time within the contracted period. This paper addresses the problem of optimal designing uptime-guarantee maintenance contracts for the service providers.

This study is particularly motivated by the contracting practices in high-tech medical equipment industry such as medical imaging and addresses an ongoing problem at a major European manufacturer. The annual maintenance service cost for medical imaging can be as much as 8.5 percent of the initial purchase cost of the equipment and it has become a key competitive factor among different manufacturers (Sferrella, 2012). Not surprisingly, specific patterns of servitization can be observed among manufacturers of medical technology (Schröter and Lay, 2014). Contractual commitments on uptime are quite common with medical equipment (Mancino and Siachos, 2007). However, Cruz and Rincon (2012) provide evidence that research into the outsourcing of maintenance services in health care sector is still in its infancy stages.

Among all the terms and conditions of an uptime-guarantee contract, we focus on two main features: the guaranteed uptime level, and the price of the contract. Shifting from cost-based to value-based contract pricing approach, we assume that the service provider's goal in designing an uptime-guarantee contract is to maximize its (expected) profit. The key for optimal design of such contracts is to understand the customer's valuation for having the equipment up and running per unit of working time which is the basis for customer's decision regarding purchasing an uptime-guarantee contract or relying solely on corrective maintenance. Charging too high a price for a contract means that customers would rather not accepting the contract while charging too low a price leaves the service provider at loss. However, in almost all real life situations service providers do not know the exact customers' valuations. Overcoming the latter hurdle in designing an uptime-guarantee contract is central in this paper. In order to tackle this issue we take a black box approach to the cost of maintenance services offered by the service provider. The corresponding costs can be estimated with analytical models, simulation or other techniques as studied extensively in the literature.

In this paper, we solve the problem of optimal design of uptime-guarantee contracts for a service provider who does not know the customers' valuation for revenue obtained by an additional percentage of uptime of the contracted systems. We construct a Stackelberg game to model and analyze the dynamics between the service provider and the customer. The service provider acts as the leader by announcing the contract or the menu of contracts. The customer makes its decision regarding purchasing contract accordingly by evaluating its (expected) improvement in revenue under the contract. Although the customer's exact valuation is unknown, we assume that the service provider knowns its probability distribution. As a technical requirement for tractability we assume that the customer's valuation has an Increasing Generalized Failure Rate (IGFR) distribution function. Many well-known distribution functions satisfy the IGFR condition, e.g., exponential, normal, logistic, Weibull, gamma, beta, Cauchy, etc. (Banciu and Mirchandani, 2013). The IGFR distribution assumption is common in supply chain revenue management and pricing literature (Lariviere, 2006). The IGFR assumption in our situation ensures that the service provider's profit function is unimodal and has a unique maximizer.

In the first part of this paper, we consider the case where the service provider offers a single contract to the customer. We specifically investigate the contracts which are interesting from the perspective of both customer and service provider. That is, we look for contracts which have a positive chance of being purchased by the customer and generates positive profit for the service provider. As we show, the existence of such contracts can be ensured by requiring a condition on the contract's guaranteed uptime level. We then develop a closed-form formula for the optimal price of contract with the given guaranteed uptime. With discrete choice set for contracts guaranteed uptime levels, the optimal price formula allows one to examine all the options to find the most profitable guaranteed uptime level.

The second part of the paper extends the analysis to situations where the service provider offers menus of multiple contracts to the customer. Contract menus enable the service provider to extract more profit from a customer that actually has high valuation without risking the potential profit that can be obtained from a customer with low valuation. Extending the notion of desirable contracts to contract menus, we seek to optimize the prices of the different contracts in menu assuming that their guaranteed uptime levels are fixed a priori. To carry out the analysis in this case we make the additional assumption that the costs associated with proving maintenance contracts with different guaranteed uptime levels are taken from an increasing and convex function. The latter assumption supports the practical cases where guaranteeing higher uptime levels are increasingly costly. We provide necessary and sufficient conditions on the vector of guaranteed uptime levels in a contract menu that ensures the existence of desirable contract menus. Accordingly, we provide a sequential method for finding the optimal prices of the contracts contained in a contract menu. A key observation is that optimal pricing of a contract menu is essentially the same as optimizing the single contracts it includes according to increasing order of their guaranteed uptime levels.

The rest of this paper is organized as following. Section 2 overviews the previous relevant studies in the literature. Section 3 outlines the elements of the mathematical model and elaborates on a special property of IGFR distribution functions which is used in the paper. The design of singular contract discussed in Section 4. The analysis to contract menus is carried out in Section 5. Section 6 concludes the paper.

2 Literature Review

We overview the extant work on multi-agent maintenance management where more than one decision maker is involved in the analysis. The literature covers a multitude of aspects related to decentralized maintenance management and exploit an array of methods to obtain insights with regard to the relationships between the different parties involved in contractual maintenance agreements.

Some of the papers in the literature focus on the decision making problems faced by the agents involved in maintenance contracts. In a multi-criteria decision making framework, de Almeida (2001) investigates the optimal choice among a set of contracts incorporating their risk and costs and the consequences on the performance of the equipment. Wang (2010) studies three types of contracts between a service provider and an equipment owner based on the extent of maintenance services provided while emphasizing on the relation between inspections and repair services. Assuming fix contract prices, the last paper analyzes the equipment owner's choice of contract and its relationship with the inspection intervals offered by the service provider.

Several papers in the literature study the optimal pricing of service contracts drawing upon the framework of Stackelberg games under the critical assumption and all the information regarding various aspects of the system is commonly known by the agents. Murthy and Yeung (1995) discuss the optimal strategies for both equipment owner and service provider in a setting where maintenance services are given in two types: pre-planned and immediate. The optimal contract prices are calculated based on customer's known revenue for a unit time of workable equipment time. Murthy and Asgharizadeh (1999) and Ashgarizadeh and Murthy (2000) study the optimal decisions of one or more equipment owners and a service provider in terms of right choice of contract, contract prices, and service channels. Rinsaka and Sandoh (2006) extend the work of Murthy and Asgharizadeh (1999) to the time after the initial warranty period is expired. Within a multi-stage decision making framework, Hartman and Laksana (2009) examine the optimal strategies for equipment owners in terms of right choice of extended warranty contract as well as the service provider's profit maximizing contract prices. They showed that by offering multiple contracts the service provider can increase its profit. Tong et al. (2014) discuss pricing strategies for a provider of two dimensional warranty contracts. Esmaeili et al. (2014) study the decision making process with regard to the choice of contract and its attributes in a three-level warranty service contract among manufacturer, service provider and customer. All of these papers assume that the equipment owner's revenue as the result of utilizing the equipment is known—the condition which we relax in this paper.

Although Stackelberg game is the most common game theoretical approach taken in the literature, few authors use cooperative and/or bargaining games to study the design of maintenance contracts. Hamidi et al. (2014) analyze a two-player cooperative game between an equipment owner, who announces the time of replacement of a part, and the service provider, who chooses the order time of the part. Jackson and Pascual (2008) consider a negotiation scenario where the optimal contract prices, preventive maintenance intervals, and response times are set to equally divide the profits among an equipment owner and a service provider.

Another stream of research on maintenance contract design deals with the fact that the service provider is not entirely aware of all customer's attributes. Taking into account the indeterministic nature of customers' attitude to risk, Huber and Spinler (2012) formulate a model which allows the service provider to manage its revenue by setting the prices of its full-service, or on-call maintenance service contracts. They provide formulations for service provider's cost structure and give a closed form formula for the optimal contract price under the assumption that the customers' attitude toward risk is uniformly distributed. Huber and Spinler (2014) extend the latter model to account for learning, optimized maintenance, and information asymmetry between the customers and service provider. In this paper, we obtain the optimal prices based on anticipation for customers' valuation of obtainable revenue when the equipment is in workable conditions, give the customers the option not to sign a contract with the service provider, and outline the optimal prices for a wide range of distributions functions. Moreover, we determine the optimal prices for menus of contracts.

An alternative approach to the relationship between the agents seeks to optimize the system-wide profit by coordinating the efforts of the parties involved—as opposed to focusing on either the customer's or the service provider's decision making problems. Within a deterministic framework, Tarakci et al. (2006a) and Tarakci et al. (2006b) analyze several mechanisms, including a pricing scheme for maintenance contracts, to make the optimal intervals for preventive maintenance from the perspectives of both service provider and equipment owner coincides. Tseng and Yeh (2013) extend the single processor case discussed in Tarakci et al. (2006a) for risk-averse customers.

Instead of relying on game theory to analyze the interactions between the agents involved, some authors use models that reflect demand elasticity to capture the effects of contracts with different terms and conditions of the service provider's profit. Drawing upon a non-linear deterministic demand function to approximate customer's sensitivity in price and delivery time, So and Song (1998) propose a mathematical model to study the interrelations among pricing, delivery time guarantee and capacity expansion decisions of a service provider. In a holistic approach to optimize contract prices and uptime levels, as well as network of maintenance facilities and personnel, Lieckens et al. (2015) use a multinomial logit model to estimate the probabilities of customers choosing among full-service, on-call, or no contract options. They indirectly obtain contract choice probabilities as functions of price elasticity and downtime sensitivity of customers.

3 Mathematical Model

The problem we consider in this paper comprises an Original Equipment Manufacturer (OEM), hereafter the service provider (he), who offers maintenance service contracts with guaranteed uptime levels to an equipment owner, hereafter the customer (she). The uptime level d, $0 \leq d \leq 1$, of an equipment represents the percentage of working time during a specific time period, usually a year, which the equipment is in perfect operational conditions.

A guaranteed uptime level d reflects the service provider's commitment toward maintaining the equipment so that it has an uptime level of d or higher during the contracted period. For providing a contract with a guaranteed uptime level, the service provider incurs different types of costs—including preventive and corrective maintenance costs as well as penalties associated with failing to maintain the guaranteed uptime. Although such costs are indeterministic in nature, we assume that the service provider at least knowns their expected amounts. We abstract such costs and let c denote the expected total cost of a contract with guaranteed uptime d. We denote a single uptime-guarantee contract with $t = (p, d)$ where p is the price of the contract t and d is its guaranteed uptime level. The choices for different uptime levels to be offered to the customer forms a finite discrete set.

We assume that without purchasing an uptime-guarantee contract and only relying on corrective maintenance services of the service provider, the customer is able to attain an expected uptime level of d_0. The base uptime level d_0 is also known by the service provider. The expected costs associated with corrective maintenance services—including parts, personnel, etc.—is denoted by c_0. Naturally, the guaranteed uptime level offered by the service provider must be higher than the base uptime level that the customer expects under corrective maintenance, i.e., $d > d_0$. Also, guaranteeing an uptime level higher than the base level involves higher costs thus we assume $c > c_0$.

The customer has a certain *valuation* for having the equipment up and running in the planning period. We let v be the customer's valuation for one percent increase in expected uptime. The valuation v can be interpreted as the revenue that the customer can generate during this additional working time. To extract the most profit out of a contract, the service provider must consider the value that the contract generates for the customer. Although the service provider is unaware of the exact customer's valuation, he knowns its probability distribution. We assume that the customer's valuation posses a density function f with its support on $V = [0, v_{\max}]$ and is differentiable at all interior points of V. The probability distribution is denoted with F and its complement with $\overline{F}$. Furthermore, we assume that the customer's valuation has an Increasing Generalized Failure Rate (IGFR) distribution. We elaborate on this assumption below.

3.1 IGFR distributions

A probability distribution F satisfies the IGFR condition if $xf(x)/\overline{F}(x)$ is non-decreasing (or weakly increasing) everywhere in its support such that $F(x) < 1$ (Lariviere, 2006). The family of IGFR distributions includes exponential, normal, log-normal, logistic, Weibull, gamma, beta, Cauchy, etc. (Banciu and Mirchandani, 2013).

The following lemma presents a technical property of IGFR distribution functions which would be used later in this paper.

Lemma 1. *Let F be an IGFR distribution. Define $G(x) = \overline{F}(x) - (x-a)f(x)$ with $a > 0$ and suppose $x^* = G^{-1}(0)$ exists such that $x^* > a$. We have $dG(x)/dx|_{x^*} < 0$.*

Proof. By IGFR assumption, $xf(x)/\overline{F}(x)$ is non-decreasing, that is,

$$\frac{d\frac{xf(x)}{\overline{F}(x)}}{dx} = \frac{f(x)}{\overline{F}(x)} + \frac{x\frac{df(x)}{dx}}{\overline{F}(x)} + x\frac{f^2(x)}{\overline{F}^2(x)} \geq 0.$$

Equivalently, we have,

$$-\frac{df(x)}{dx} \leq f(x)\frac{\overline{F}(x) + xf(x)}{x\overline{F}(x)}.$$

Multiplying both sides by $x - a$, assuming that $x > a$, and subtracting $2f(x)$ yields

$$-2f(x) - (x-a)\frac{df(x)}{dx} = \frac{dG(x)}{dx} \leq f(x)\frac{-2x\overline{F}(x) + (x-a)\overline{F}(x) + x(x-a)f(x)}{x\overline{F}(x)}.$$

If $x^* = G^{-1}(0)$ exists such that $x^* > a$, then it must be that $(x^* - a)f(x^*) = \overline{F}(x^*)$. Thus, last inequality obtains

$$\left.\frac{dG(x)}{dx}\right|_{x=x^*} \leq f(x^*)\frac{-2x^*\overline{F}(x^*) + (x^* - a)\overline{F}(x^*) + x^*\overline{F}(x^*)}{x^*\overline{F}(x^*)} = f(x^*)\frac{-a}{x^*}.$$

Since $f(x) > 0$ for all $x \in V$ we conclude that $dG(x)/dx|_{x^*} < 0$. □

Lemma 1 shows that for IGFR distributions, the critical point of the function G happening after a is a maximum.

4 Singular Contracts

In this section we study the situation where the service provider offers a single contract to the customer. We start by stating the customer's and service provider's individual problems. Afterwards, we draw upon a Stackleberg game to combine the two problems. We tackle the contract design problem by optimizing the price of a contract subject to a given uptime-guarantee level. In the final step, we comment on the procedure to find the most profitable guaranteed uptime levels.

4.1 Customer's profit

The customer is a rational decision maker who would purchase an uptime-guarantee contract to improve its (expected) profit. In doing so, the customer compares its profit under an uptime-guarantee contract with that in situation of operating without a uptime-guarantee contract and relying only on corrective maintenance. The profit function of the customer under the choice of an uptime-guarantee contract $t = (p, d)$ is

$$u(v, t) = v(d - d_0) - (p - c_0). \tag{1}$$

Remember that d_0 and c_0 are the expected uptime and its associated cost in case of no uptime-guarantee contract. Thus, $u(v, t)$ is the additional profit obtained in a period as the result of choosing t.

The customer will certainly purchase the contract t if $u(v, t) > 0$, i.e., the contract results in positive profit. Therefore, a customer with a valuation v would purchase the contract whenever her valuation is higher than the threshold $v_0 = (p - c_0)/(d - d_0)$. In case $v = v_0$, which implies $u(v, t) = 0$, purchasing the contract has no additional benefit and the customer is indifferent between acquiring and not acquiring the contract. If this is the case, the customer may or may not purchase the contract. Figure 1 illustrates the customer's profit under contract t and the interval of valuation where in the contract would be purchased.

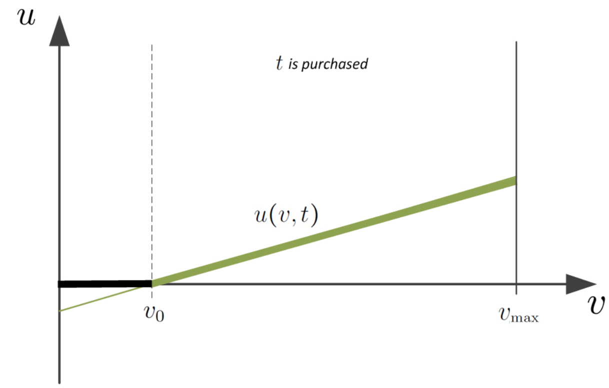

Figure 1: The customer's profit under a single contract

4.2 Service provider's profit

The service provider can expect two outcomes by offering the uptime-guarantee contract $t = (p, d)$. Either the customer refrain from purchasing the contact or the contract would be purchased. If the contract is not purchased then the profit of zero would be obtained by the service provider. Otherwise, if the contract is purchased, the service provider obtains the profit of $p - c$. Remember that c is the expected cost involved in upholding the contract with guaranteed uptime level d.

4.3 Singular contract design problem

In this section we address the following problem faced by the service provider: what is the price p that maximizes the expected profit of a contract with the guaranteed uptime level d? To tackle the pricing problem, the profit functions of the service provider and the customer must be considered simultaneously. This situation can be handled by a Stackelberg game where the service provider, acting as the leader, proposes the contract in anticipation of optimal response of the customer who takes on the role of follower.

Assuming that the service provider is risk-neutral, we can formulate his expected profit as

$$\Pi(t) = (p - c)Pr\{v|t \text{ is purchased}\}. \tag{2}$$

The expected profit of the service provider upon offering the contract t is the associated profit of this contract multiplied by the probability of the customer's valuation be such the she purchases the contract. As seen in Figure 1, contract t would always be purchased whenever $v > v_0$ and it may be purchased when $v = v_0$. Thus, in case $v_0 \leq v_{\max}$, the probability of contract being purchased is $\overline{F}(v_0)$. Note that with a continuous probability function the customer's randomization at $v = v_0$ is irrelevant. By combining the profit functions of the customer and the service provider we can denote the expected profit of the service provider, taking into account the customer's response, in the following manner:

$$\Pi(t) = \begin{cases} (p-c)\overline{F}\left(\frac{p-c_0}{d-d_0}\right) & \text{if } 0 \le \frac{p-c_0}{d-d_0} < v_{\max} \\ 0 & \text{if } \frac{p-c_0}{d-d_0} \ge v_{\max} \end{cases}. \tag{3}$$

In above formulation we used the fact that the probability of a continuous random variable being equal to a single value is zero.

4.4 Desirable contracts

Not all possible contracts are worth considering. In this section we define two desirable properties for uptime-guarantee contracts that capture the customer's as well as service provider's expectation of a contract.

Definition 1. *The contract* $t = (p, d)$ *is* essential *if* $(p - c_0)/(d - d_0) < v_{\max}$.

If the price of the contract is set such that $v_0 = (p - c_0)/(d - d_0) \ge v_{\max}$, there is no positive chance that the customer purchases the contract. Thus, the first condition reflects the requirement that an offered contract must have a positive chance of being selected.

Definition 2. *The contract* $t = (p, d)$ *is* profitable *if* $p - c > 0$.

A profitable contract results in positive profit for the service provider. Therefore, from the perspective of the service provider, a contract is only worthwhile to offer if it is profitable. Since $c > c_0$, profitability of a contract also means that $(p - c_0)/(d - d_0) > 0$.

From equation (3) it can be verified that with an essential and profitable contract, the service provider's profit boils down to:

$$\Pi(t) = (p-c)\overline{F}\left(\frac{p-c_0}{d-d_0}\right). \tag{4}$$

4.5 Optimal single contracts

In order to obtain the optimal price of an essential and profitable contract which guarantees the uptime level of d, we must solve the following problem:

$$\max_{p} \quad (p-c)\overline{F}\left(\frac{p-c_0}{d-d_0}\right) \tag{5}$$

$$s.t. \quad \frac{p-c_0}{d-d_0} < v_{\max} \tag{6}$$

$$p - c > 0 \tag{7}$$

In the above program, constraint (6) ensures that the obtained contract is essential while constraint (7) ascertains its profitability. When these two conditions are met, the objective function maximizes the service provider's expected profit.

Before solving the optimization problem above which optimizes the service provider's expected profit given a guaranteed level d, we express a condition for d.

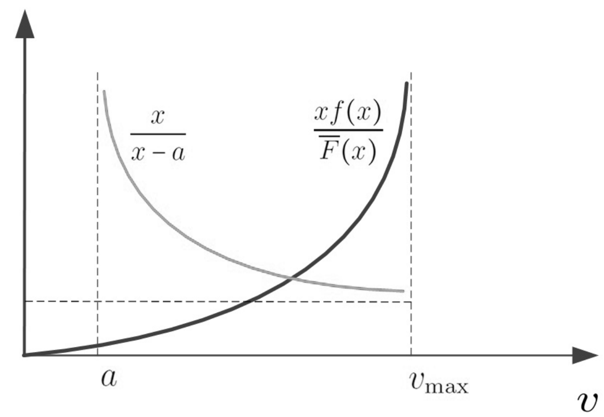

Figure 2: Functions used in proof of Theorem 1

Definition 3. *The guaranteed uptime level d is* admissible *if* $(c-c_0)/(d-d_0) < v_{\max}$.

The amount $(c-c_0)/(d-d_0)$ is the lowest valuation for the customer who would purchase a contract with guaranteed service level d which leaves the service provider with no loss. The admissibility requirement on guaranteed uptime levels is needed to ensure that the associated contracts are essential and profitable. The latter is formalized in the next lemma.

Lemma 2. *A necessary condition for the existence of an essential and profitable contract with guaranteed uptime level d is that d be admissible.*

Proof. Suppose the contrary, that is, $(c-c_0)/(d-d_0) \geq v_{\max}$, and there exists a profitable and essential contract $t = (p, d)$. The profitability of (p, d) implies that $p > c$ which in turn results in having $(p-c_0)/(d-d_0) > (c-c_0)/(d-d_0)$. By the contrary assumption we conclude that $(p-c_0)/(d-d_0) \geq v_{\max}$. However, the latter contradicts t being essential. Therefore, to have an essential and profitable contract d must be admissible. □

We are now ready to present the main result of this section regarding the optimal prices of essential and profitable uptime-guarantee contracts.

Theorem 1. *For an admissible guaranteed uptime level d, there exists a unique optimal price p^* such that the contract (p^*, d) is essential, profitable, and maximizes the service provider's expected profit. The optimal price p^* is obtained from the following implicit condition:*

$$\overline{F}\left(\frac{p^*-c_0}{d-d_0}\right) = \frac{p^*-c}{d-d_0} f\left(\frac{p^*-c_0}{d-d_0}\right). \tag{8}$$

Proof. Consider the relaxation of the program in (5)-(7). The first derivative of the objective function in (5) with respect to p can be written as $\overline{F}(x) - (x-a)f(x)$, where $x = (p-c_0)/(d-d_0)$ and $a = (c-c_0)/(d-d_0)$. The critical point, upon existence, happens wherever two functions $xf(x)/\overline{F}(x)$ and $x/(x-a)$ intersect such that $x > 0$.

The IGFR assumption implies that $xf(x)/\overline{F}(x)$ is non-decreasing for every $x \in V$ such that $\overline{F}(x) > 0$. Due to properties of distribution functions we know that $xf(x)$ is bounded and we can find $x \in V$ such that $\overline{F}(x)$ is arbitrary close to zero. Thus $xf(x)/\overline{F}(x)$ has the range $[0, \infty)$. On the other hand, the function $x/(x-a)$ is strictly decreasing for $x > a$ and has the range $(1, \infty)$. Note that with an admissible d it is the case that $a < v_{\max}$. Thus, there must exist a unique $a < x^* < v_{\max}$ such that $x^* f(x^*)/\overline{F}(x^*) = x^*/(x^* - a)$ which implies that $\overline{F}(x^*) - (x^* - a) f(x^*) = 0$ (see Figure 2). By lemma 1 it follows that the second derivative at x^* is negative so x^* is the unique maximum.

To complete the proof, we show that the constraints in (6) and (7) hold. Since $x^* < v_{\max}$ we have $(p^* - c_0)/(d - d_0) < v_{\max}$ thus (6) holds, and as $x^* > a$ we have $p^* - c > 0$ thus (7) holds. □

According to the condition in (8), the optimal price of a contract is such that the probability of the customer's choosing the contract equals $v_0 f(v_0)$.

Theorem 1 shows that if d is admissible, one can come up with an optimal contract which is essential and profitable. In combination with Lemma 2 which states a necessary condition for existence of essential and profitable single contracts, the next observation follows immediately.

Corollary 1. *Let d be a guaranteed uptime level. A unique essential and profitable uptime-guaranteed contract with the highest expected profit for the service provider exists if and only if d is admissible.*

After solving the price optimization problem, one needs to address the problem of finding the guaranteed uptime level which leads to the contract that has the highest expected profit. The options for different uptime levels constitute a discrete choice space since in reality the guaranteed uptime contracts are often vary by degrees of one percent. In this case, the most profitable guaranteed uptime level can be easily found by comparing the maximum expected profits of contracts associated with all possible options for uptime levels.

5 Contract Menus

In this section we extend our analysis to study situations where the service provider offers a collection of contracts, i.e., a *contract menu*, to the customer. The service provider's incentive for doing so is to take advantage of the different possible valuations of the customer to get the most out of a high-valuation customer who is willing to pay more to purchase contracts with higher guaranteed uptime levels while avoiding the risk of loosing a low-valuation customer.

Figure 3 illustrates the logic of contract menus. As seen in this figure, there are three contract in the contract menu. Depending on the customer's valuation, there are different regions where a contract is the best choice. While the cheaper contract with lower guaranteed uptime level would be purchased by the customer who has low valuations, more expensive contracts with higher guaranteed uptime levels will be purchased by the customer whose valuation is closer the upper bound.

Formally, a contract menu is a non-empty set of single contracts denoted by $T = \{t_1, ..., t_m\}$ where for $1 \le k \le m$, $t_k = (d_k, p_k)$ is referred to as contract k. We assume hereafter that the indices of contracts in a contract menu are arranged according to increasing order of uptime

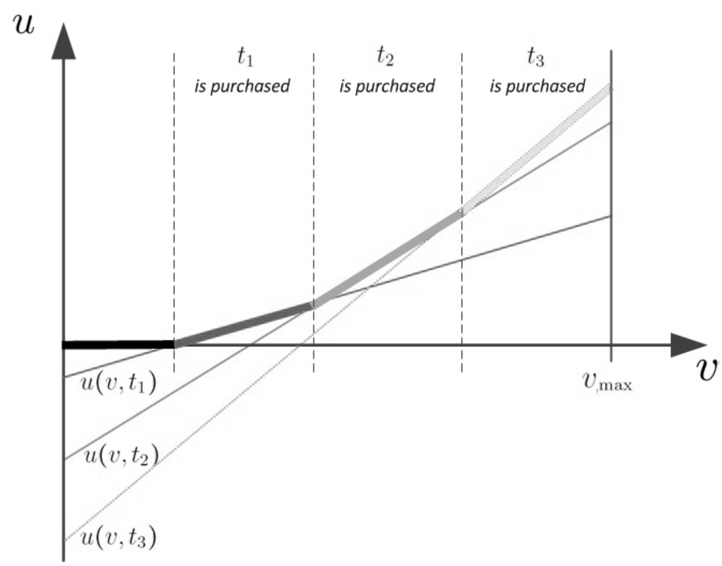

Figure 3: The customer's profit under a contract menu

levels, i.e. that for $k = 1, ..., m-1$ we have $d_k < d_{k+1}$. To be able to compare the choice of contracts with the choice of no contract, we define the dummy contract $t_0 = (d_0, p_0)$ with $p_0 = c_0$. This contract represents the corrective-maintenance-only services. Let $T^+ = T \cup t_0$ be the augmented contract menu including the dummy contract.

We make the additional assumption in this section that the costs associated with providing uptime-guarantee contracts with different levels are points on an increasing and convex function. This means that the slope of the line segment connecting two points (c_k, p_k) and (c_{k+1}, p_{k+1}) increases as k get larger. Thus, the rate of growth of costs associated with providing contracts which guarantee higher uptime levels is increasing.

5.1 Customer's profit

In line with equation (1), the profit to a customer under contract $t_k \in T^+$ can be formulated as

$$u(v, t_k) = v(d_k - d_0) - (p_k - c_0).$$

The customer will certainly purchase a contract in T if for some $1 \leq k \leq m$ it holds that $u(v, t_k) > 0$. Otherwise the customer would either be indifferent between purchasing or not purchasing a contract, or he would be better off without a contract. The rational customer maximize her profit by choosing the best contract in the augmented contract menu T^+. For a customer with the valuation v, the contract t_k is an optimal choice if $u(t_k, v) \geq u(t_{k'}, v)$ for all $t_{k'} \in T^+$. In case there are multiple optimal contracts, the customer chooses a contract among the set of optimal contracts randomly. We show later that the actual randomization among different optimal contracts does not have any effect on service provider's expected profit.

The linearity of customer's profit functions enables straightforward comparisons among different contracts in a contract menu. In fact, one can calculate thresholds for every pair of contracts wherein customer's preference for that pair of contracts switches at the threshold. Let T be a contract menu with $m \geq 1$. For $t_k, t_{k'} \in T^+$ such that $k < k'$ we define *pivot of* k *and* k' by

$$v_{k,k'} = \frac{p_{k'} - p_k}{d_{k'} - d_k}. \tag{9}$$

It is straightforward to see that the customer with valuation v prefers contract k' to k whenever $v > v_{k,k'}$ since $u(v,k') > u(v,k)$. The contract k would be chosen over contract k', i.e. $u(v,k') < u(v,k)$, whenever $v < v_{k,k'}$. In case $v = v_{k,k'}$, we have $u(v,k') = u(v,k)$ which implies that the customer is indifferent between k and k' and chooses randomly between the two.

5.2 Service provider's profit

The service provider's profit can be formulated in the same way as in singular contracts. To provide the contract t_k, the service provider incurs cost c_k. If the price for this contract is set to p_k, then the profit to the service provider as the result of customer's accepting this contract is $p_k - c_k$. If $k = 0$ is the choice of a customer, then the profit of zero would be obtained.

5.3 Contract menu design problem

The service provider, acting as the Stackelberg leader, could foresee the exact choice of the customer if her valuation was known. Without the exact knowledge on the customer's valuation, the service provider's problem is to set the prices of contracts in the menu so as to maximize his expected profit. We take the similar approach as in the case of singular contracts and assume that the service provider chooses the number of contracts in a menu and their guaranteed uptime levels according to its strategic plans. In this way we focus on the problem of maximizing service provider's expected profit by setting the prices of different contracts.

The formulation of service provider's expected profit, taking into consideration the customer's reactions, is more complicated under contract menus. Nevertheless, we can express the latter in terms of the probabilities of different contracts being purchased by the customer multiplied by the profit of corresponding contracts for the service provider in the following manner:

$$\Pi(T) = \sum_{k \in T} (p_k - c_k) Pr\{v | t_k \text{ is purchased}\}. \tag{10}$$

In the next section we use the notion of desirable contracts to attain a closed-form formulation of the service provider's expected profit as the function of contract prices.

5.4 Desirable contract menus

Here we extend the notions of essential and profitable contract menus by following the same logic as in the case of singular contracts. The definition of an essential contract menu mandates that all contracts in the menu have a positive chance of being purchased by the customer so that no contract is redundant.

Definition 4. *The contract menu T is essential if for every $t_k \in T$, $Pr\{v|t_k$ is purchased$\} > 0$.*

We can express the required condition of an essential contracts in terms of intervals of valuations. First, note that for a contract in a contract menu to have a chance of being purchased, there must exists $v \in V$ such that for all $t_{k'} \in T^+$ we have $u(t_k, v) \geq u(t_{k'}, v)$. Second, with a continuous distribution the probability of the customer's valuation being a single point is zero. Therefore, in order for a contract menu to be essential, the first condition above must hold for every single contract of the menu over a non-empty interval of valuations, i.e., an interval whose boundary points are non-identical. Since the valuation distribution function has a positive value everywhere on its support, the latter ensures that the probability of a contract being purchased is positive. We express the latter formally in the following corollary.

Corollary 2. *Let T be a contract menu and $t_k \in T$. We have $Pr\{v|t_k$ is purchased$\} > 0$ if and only if there exists a non-empty interval $V_k \subseteq V$ such that for every $v \in V_k$ we have $u(t_k, v) \geq u(t_{k'}, v)$ for all $t_{k'} \in T^+$.*

As we develop the pricing formula only for essential contract menus, it is vital to have a complete characterization of such contracts. The next observation provides the initial step in this regard.

Lemma 3. *For a contract menu T of size $m \geq 2$ to be essential, the following conditions are necessary: (i) $v_{0,1} < v_{1,2} < \ldots < v_{m-1,m}$, (ii) $v_{1,2} > 0$, (iii) $v_{m-1,m} < v_{\max}$.*

Proof. (i) Consider $l, h, k \in \{0, \ldots, m\}$ such that $l < h < k$. Suppose the contrary, and assume T is essential and $v_{h,k} \leq v_{l,h}$. We continue in two steps.

(Step I) We show if $v_{h,k} \leq v_{l,h}$, then it must be that $v_{h,k} \leq v_{l,k} \leq v_{l,h}$. Let $A = [0, v_{h,k}]$, $B = [v_{h,k}, v_{l,h}]$, and $C = [v_{l,h}, v_{\max}]$. Considering the property of pivot points we have:

$v \in A$	$v \in B$	$v \in C$
$u(v,h) \geq u(v,k)$	$u(v,h) \leq u(v,k)$	$u(v,h) \leq u(v,k)$
$u(v,l) \geq u(v,h)$	$u(v,l) \geq u(v,h)$	$u(v,l) \leq u(v,h)$

From the above we can deduce that $u(v,l) \geq u(v,k)$ for all $v \in A$, and $u(v,l) \leq u(v,k)$ for all $v \in C$. Thus, it must be that $v_{l,k} \in B$.

(Step II) We show if $v_{h,k} \leq v_{l,k} \leq v_{l,h}$, then there exists no non-empty interval such that t_h is an optimal choice for the customer. Let $B_1 = [v_{h,k}, v_{l,k}]$ and $B_2 = [v_{l,k}, v_{l,h}]$. We have the following cases:

$v \in A$	$v \in B_1$	$v \in B_2$	$v \in C$
$u(v,h) \geq u(v,k)$	$u(v,h) \leq u(v,k)$	$u(v,h) \leq u(v,k)$	$u(v,h) \leq u(v,k)$
$u(v,l) \geq u(v,h)$	$u(v,l) \geq u(v,h)$	$u(v,l) \geq u(v,h)$	$u(v,l) \leq u(v,h)$
$u(v,l) \geq u(v,h)$	$\min\{(v,l),(v,k)\} \geq u(v,h)$	$\min\{(v,l),(v,k)\} \geq u(v,h)$	$u(v,h) \leq u(v,k)$

Based on the above table we understand that everywhere in V the choice of h results in a profit for the customer which is at most as high as either k or l. Considering that with different guaranteed uptime levels any two contracts can only have a single point at which their profits are equal, we conclude that contract h is among the optimal choices of the customer at most at one point in V thus there is no non-empty interval on V where h is the optimal contract. Hence, T is not an essential contract menu. This is a contradiction so it must be that $v_{h,k} > v_{l,h}$. Extending this logic for all trios of contracts in T obtains the claim in part (i).

(ii) If $v_{1,2} \leq 0$, then the customer with any valuations in $(0, v_{\max}]$ prefers contract t_2 to t_1. Thus contract t_1 is within the optimal choice set of the customer at most at a single point $v = 0$ and not in a non-empty interval in V which implies that T is not essential. Hence it must be that $0 \leq v_{1,2}$

(iii) If $v_{m-1,m} \geq v_{\max}$, then the customer with any valuations in $[0, v_{\max})$ prefers contract t_{m-1} to t_m. Thus contract t_m is at most optimal at the single point $v_{\max}$ which means that T is not essential. Hence it must be that $v_{m-1,m} < v_{\max}$. □

The first condition in Lemma 3 requires that pivots of consecutive contracts be ordered such the contracts with lower indices have their pivots before the contracts with higher indices. The second condition prescribes that the pivot of first two contracts be situated in positive real numbers. Finally, the third condition necessitates that the pivot of last two contracts happens prior to the upper bound for the customer's valuation. In the following observation, we prove that these three conditions together characterize the entire class of essential contract menus.

Lemma 4. *A contract menu T of size $m \geq 2$ is essential if and only if (i) $v_{0,1} < v_{1,2} < \ldots < v_{m-1,m}$, (ii) $v_{1,2} > 0$, and (iii) $v_{m-1,m} < v_{\max}$.*

Proof. [If part] Since (i) holds, we can define $\dot{V}_k = [v_{k-1,k}, v_{k,k+1}]$ for $1 \leq k < m-1$, and $\dot{V}_m = [v_{m-1,m}, \infty)$ in such a way that for every $1 \leq k \leq m-1$ and every $v \in \dot{V}_k$ it holds that

$$u(v, t_k) \geq u(v, t_{k'}) \text{ for every } 1 \leq k' \leq m-1.$$

Imposing the conditions (i) and (iii) assures that T divides V into non-empty intervals $V_1 = [0, \max\{0, v_{0,1}\}]$, $V_2 = [\max\{0, v_{0,1}\}, v_{1,2}]$, $V_k = [v_{k-1,k}, v_{k,k+1}]$ for every $3 \leq k \leq m-1$, and $V_m = [v_{m-1,m}, v_{\max}]$ where for every $v \in V_k$, $1 \leq k \leq m$, it holds that t_k is an optimal choice for the customer which ensures that T is an essential contract.

[Only-if part] By Lemma 3, conditions (i) to (iii) are necessary for T to be an essential contract. By previous part we see that when the three conditions hold, T is in fact essential. Thus, conditions (i) to (iii) together are sufficient for T to be essential. □

The service provider has sufficient incentives to offer a contract menu if it is profitable. The linearity of customer's utility function implies that if a contract t_k is such that $u(v, t_k) \geq 0$, then in case the customer has higher valuations, $v' > v$, it is also the case that $u(v', t_k) \geq 0$. However, the reverse is not necessarily the case. Therefore, offering multiple contracts by the service provider is only reasonable if contracts with higher guaranteed uptime levels would result in more profit for him. Otherwise, the contracts with higher guaranteed uptime levels cannibalize the profits of those with lower uptime levels.

Definition 5. *A contract menu T is profitable if (a) every contract $t_k \in T$ is profitable, and (b) for every $t_k, t_k' \in T^+$ such that $k' > k$ it holds that $p_{k'} - c_{k'} > p_k - c_k$.*

According the above definition, a contract menu is profitable if the price of every singular contract in it outweighs the cost of providing that contract, and contracts with higher indices generate more profit that the contracts with lower indices.

The expected profit of the service provider under essential and profitable contract menus can be expressed in a concise manner. The characterizing conditions of essential contract menus, along with the definition of pivots of two contracts in (9), imply that for an essential contract menu the valuation range of the customer can be partitioned into sub-intervals such that the best choice of contract for the customer in consecutive sub-intervals are consecutive contracts in T. That is, for every $v \in (v_{k-1,k}, v_{k,k+1})$, $k = 1, \ldots, m-1$, the optimal contract for the customer is t_k. The probability of customer's valuation to be in this range is $F(v_{k,k+1}) - F(v_{k-1,k})$. Also, for every $v \in (v_{m-1,m}, v_{\max})$, the optimal contract is t_m and the probability of having a customer whose valuation is within this interval is $\overline{F}(v_{m-1,m})$. Note that at the boundary points of the intervals the customer would chooses randomly between the corresponding optimal contracts. However, when calculating the expected profit of the service provider, continuity of the valuation distribution function makes the consideration of such points unnecessary. Given an essential and profitable contract menu T of size m, the service provider's expected profit in (10) can be written as

$$\Pi(T) = \sum_{k=1}^{m-1} (p_k - c_k) \left[F(v_{k,k+1}) - F(v_{k-1,k}) \right] + (p_m - c_m) \overline{F}(v_{m-1,m}). \tag{11}$$

5.5 Optimal contract menus

The problem we solve in this section is the following: given a vector of guaranteed uptime levels d_T associated with the contract menu T, find the corresponding price vector p_T such that T is essential, profitable, and has the maximum expected profit for the service provider.

We would like to maximize $\Pi(T)$, by finding the appropriate prices, while keeping the conditions of essential and profitable contract menus in place. Thus, the optimization problem to be solved is

$$\max_{p_T} \quad \sum_{k=1}^{m-1} (p_k - c_k) \left[F\left(\frac{p_{k+1} - p_k}{d_{k+1} - d_k}\right) - F\left(\frac{p_k - p_{k-1}}{d_k - d_{k-1}}\right) \right] + (p_m - c_m)\right] \overline{F}\left(\frac{p_m - p_{m-1}}{d_m - d_{m-1}}\right) \tag{12}$$

$$s.t. \quad \frac{p_{k+1} - p_k}{d_{k+1} - d_k} > \frac{p_k - p_{k-1}}{d_k - d_{k-1}} \qquad \forall k = 1, ..., m-1 \tag{13}$$

$$\frac{p_m - p_{m-1}}{d_m - d_{m-1}} < v_{\max} \tag{14}$$

$$p_k - c_k > p_{k-1} - c_{k-1} \qquad \forall k = 1, ..., m \tag{15}$$

The family of constraints in (15) ascertains that the obtained contract menu is profitable. On the other hand, constraints in (13) and (14) correspond to the conditions (i) and (iii) for essential contract menus as stated in Lemma 4. The condition (ii) is implied by constraints in (15) for $k = 2$, i.e., $p_2 - p_1 > c_2 - c_1$, which due to the increasing property of maintenance costs obtains $p_2 - p_1 > 0$. Collectively, the constraints in the above program ensure that contract menu T is essential and profitable.

The preliminary step in solving the program above is to ensure that it has at least a feasible solution. As the optimal contract prices for a contract menu subject are obtained given vectors of guarantee uptime levels, we initially introduce a condition on them.

Definition 6. *Contract menu T with $m \geq 2$ has an admissible vector of guaranteed uptime levels if $(c_m - c_{m-1})/(d_m - d_{m-1}) < v_{\max}$.*

The value $(c_m - c_{m-1})/(d_m - d_{m-1})$ represents the slope of the line segment connecting the points (c_{m-1}, d_{m-1}) and (c_m, d_m). Thus, the admissibility condition of contract menus requires that the slope of the line segment between the cost-uptime points corresponding to the last two contracts in the menu be less that the upper bound of the customer's valuation range. As we show in the following lemma, the admissibility condition of a vector of guaranteed uptime levels is a necessary requirement for existence of an essential and profitable contract menu with that uptime levels.

Lemma 5. *A necessary condition for a contract menu T to be essential and profitable is that d_T be an admissible vector of guaranteed uptime levels.*

Proof. Suppose the contrary, that is, T is an essential and profitable contract menu and d_T is not admissible, i.e.,

$$\frac{c_m - c_{m-1}}{d_m - d_{m-1}} \geq v_{\max}. \tag{16}$$

In order for a contract menu T to be profitable it must hold that $p_m - c_m > p_{m-1} - c_{m-1}$. This condition can be rewritten as

$$\frac{p_m - p_{m-1}}{d_m - d_{m-1}} > \frac{c_m - c_{m-1}}{d_m - d_{m-1}}. \tag{17}$$

Combining inequalities (16) and (17) we get $(p_m - p_{m-1})/(d_m - d_{m-1}) > v_{\max}$. The latter violates the condition (iii) in Lemma 4 which must hold for every essential contract menu. This is a contradiction as we assumed that T is an essential contract menu. Therefore, the claim of the lemma holds. □

In order to solve the non-linear and constrained optimization program in (12)–(15), we take the following approach. Considering that we are given with a vector of admissible guaranteed uptime levels, we first relax the optimization program and optimize its unconstrained version. Afterwards, we check if the obtained solutions satisfy the corresponding constraints. The relaxation of (12) can be optimized using multi-variable calculus. The following lemma highlights the critical point of the function $\Pi(T)$.

Lemma 6. *Let d_T be an admissible vector of guaranteed uptime levels. The following system yields a unique critical point for $\Pi(T)$. Set $p_0^* = c_0$ and for $1 \le k \le m$:*

$$\overline{F}\left(\frac{p_k^* - p_{k-1}^*}{d_k - d_{k-1}}\right) = \frac{p_k^* - c_k - p_{k-1}^* + c_{k-1}}{d_k - d_{k-1}} f\left(\frac{p_k^* - p_{k-1}^*}{d_k - d_{k-1}}\right). \tag{18}$$

Proof. For $1 \le k \le m-1$ the first degree derivatives of (12) are of the form:

$$\begin{aligned}\frac{\delta\Pi}{\delta p_k} = \quad & F\left(\frac{p_{k+1} - p_k}{d_{k+1} - d_k}\right) + \frac{p_{k+1} - c_{k+1} - p_k + c_k}{d_{k+1} - d_k} f\left(\frac{p_{k+1} - p_k}{d_{k+1} - d_k}\right) \\ & - F\left(\frac{p_k - p_{k-1}}{d_k - d_{k-1}}\right) - \frac{p_k - c_k - p_{k-1} + c_{k-1}}{d_k - d_{k-1}} f\left(\frac{p_k - p_{k-1}}{d_k - d_{k-1}}\right)\end{aligned}$$

and for $k = m$ we have,

$$\frac{\delta\Pi}{\delta p_m} = \overline{F}\left(\frac{p_m - p_{m-1}}{d_m - d_{m-1}}\right) - \frac{p_m - c_m - p_{m-1} + c_{m-1}}{d_m - d_{m-1}} f\left(\frac{p_m - p_{m-1}}{d_m - d_{m-1}}\right)$$

A critical point p_T^*, upon existence, is a solution to the system of equations $\{\delta\Pi/\delta p_k = 0, 1 \le k \le m\}$. Starting from the last equation and substituting the corresponding phrases in previous equations, it is straightforward to see that given the first degree derivatives above, the solution to this system of equations is obtained via the implicit functions expressed in (18).

To complete the proof it suffices to show that p_T^* exists and is unique. For $1 \le k \le m$, let $x = (p_{k+1} - p_k)/(d_{k+1} - d_k)$ and $a = (c_{k+1} - c_k)/(d_{k+1} - d_k)$. With an admissible d_T and under the assumption on maintenance costs, it is the case that $a < v_{\max}$. In this case, similar logic as the proof of Theorem 1 obtains that there must exist a unique $a < x^* < v_{\max}$ such that $\overline{F}(x^*) - (x^* - a) f(x^*) = 0$. □

The next observation establishes the fact that the vector of prices p_T^* corresponding to the critical point of $\Pi(T)$ obtained in (18) satisfies the constraints in the original optimization program.

Lemma 7. *The vector of prices obtained via implicit equations in (18) upon existence satisfies the constraints in (13)–(15).*

Proof. To show (13), fix $k \in \{0, ..., m-1\}$ and note that by optimality conditions in (18) we have

$$\frac{f\left(\frac{p_{k+1}^* - p_k^*}{d_{k+1} - d_k}\right)}{\overline{F}\left(\frac{p_{k+1}^* - p_k^*}{d_{k+1} - d_k}\right)} = \frac{1}{\frac{p_{k+1}^* - p_k^*}{d_{k+1} - d_k} - \frac{c_{k+1} - c_k}{d_{k+1} - d_k}}, \quad \text{and} \quad \frac{f\left(\frac{p_k^* - p_{k-1}^*}{d_k - d_{k-1}}\right)}{\overline{F}\left(\frac{p_k^* - p_{k-1}^*}{d_k - d_{k-1}}\right)} = \frac{1}{\frac{p_k^* - p_{k-1}^*}{d_k - d_{k-1}} - \frac{c_k - c_{k-1}}{d_k - d_{k-1}}}.$$

Suppose the contrary, that is, $\frac{p^*_{k+1}-p^*_k}{d_{k+1}-d_k} \leq \frac{p^*_k-p^*_{k-1}}{d_k-d_{k-1}}$. Then, by the IGFR assumption, it must be that

$$\frac{p^*_{k+1}-p^*_k}{d_{k+1}-d_k} - \frac{p^*_k-p^*_{k-1}}{d_k-d_{k-1}} \geq \frac{c_{k+1}-c_k}{d_{k+1}-d_k} - \frac{c_k-c_{k-1}}{d_k-d_{k-1}}$$

Since c_k for $k = 0, ..., m-1$ are taken from an increasing and convex function, it holds that $(c_{k+1}-c_k)/(d_{k+1}-d_k) > (c_k-c_{k-1})/(d_k-d_{k-1})$ which implies that $(p^*_{k+1}-p^*_k)/(d_{k+1}-d_k) > (p^*_k-p^*_{k-1})/(d_k-d_{k-1})$. This is clearly a contradiction. Therefore, it must be that $v_{k-1,k} < v_{k,k+1}$.

To show (14), note that since p^*_T exists, clearly we have $(p^*_m - p^*_{m-1})/(d_m - d_{m-1}) \leq v_{\max}$, otherwise the cdf would be undefined. It remains to show that equality does not hold. Considering the optimality condition in (18), the case $(p^*_m - p^*_{m-1})/(d_m - d_{m-1}) = v_{\max}$ implies that $f((p^*_m - p^*_{m-1})/(d_m - d_{m-1})) = 0$. However, this cannot be as $f(\cdot)$ is positive everywhere on its support.

Finally, we show that the solution satisfies constraint (15), i.e. for $1 \leq k \leq m-1$ we have $p^*_{k+1} - c_{k+1} > p^*_k - c_k$. Fix k and consider the optimality condition in (18). In this case, we would have

$$F\left(\frac{p^*_{k+1}-p^*_k}{d_{k+1}-d_k}\right) = 1 - \frac{p^*_{k+1}-c_{k+1}-p^*_k+c_k}{d_{k+1}-d_k} f\left(\frac{p^*_{k+1}-p^*_k}{d_{k+1}-d_k}\right)$$

Assume the contrary, i.e. $p^*_{k+1}-c_{k+1}-p^*_k+c_k \leq 0$. Since $f(\cdot) > 0$ for all $v \in V$, the last inequality yields $F\left(\frac{p^*_{k+1}-p^*_k}{d_{k+1}-d_k}\right) \geq 1$ which is a contradiction since $(p^*_{k+1} - p^*_k)/(d_{k+1} - d_k) < v_{\max}$ as shown in the last step. Thus, for $1 \leq k \leq m-1$ we have $p^*_{k+1} - c_{k+1} > p^*_k - c_k$. □

So far we have found a critical point of the function $\Pi(T)$ which is a feasible point for our optimization program. What remains to establish is the nature of this critical point, i.e., whether it is a minimum, a maximum, or a saddle point. In order to do so, we must check the Hessian matrix associated with $\Pi(T)$.

Lemma 8. *Let T be an essential contract menu with $m \geq 2$. For $1 \leq k \leq m$ define*

$$b_k = \frac{2}{d_{k+1}-d_k} f\left(\frac{p_{k+1}-p_k}{d_{k+1}-d_k}\right) + \frac{p_{k+1}-c_{k+1}-p_k+c_k}{d_{k+1}-d_k} \frac{\delta f\left(\frac{p_{k+1}-p_k}{d_{k+1}-d_k}\right)}{\delta p_{k+1}}.$$

The Hessian matrix associated with $\Pi(T)$ is

$$H = \begin{bmatrix} -b_1-b_2 & b_2 & 0 & 0 & \dots & 0 & 0 \\ b_2 & -b_2-b_3 & b_3 & 0 & \dots & 0 & 0 \\ 0 & b_3 & -b_3-b_4 & b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_m & -b_m \end{bmatrix} \tag{19}$$

Proof. Second degree derivatives are as below. For $1 \leq k \leq m-1$:

$$\begin{aligned}\frac{\delta^2\Pi(d_k)}{\delta p_k^2} &= -\frac{2}{d_{k+1}-d_k} f\left(\frac{p_{k+1}-p_k}{d_{k+1}-d_k}\right) - \frac{p_{k+1}-c_{k+1}-p_k+c_k}{d_{k+1}-d_k} \frac{\delta f\left(\frac{p_{k+1}-p_k}{d_{k+1}-d_k}\right)}{\delta p_{k+1}} \\ &\quad -\frac{2}{d_k-d_{k-1}} f\left(\frac{p_k-p_{k-1}}{d_k-d_{k-1}}\right) - \frac{p_k-c_k-p_{k-1}+c_{k-1}}{d_k-d_{k-1}} \frac{\delta f\left(\frac{p_k-p_{k-1}}{d_k-d_{k-1}}\right)}{\delta p_k},\end{aligned}$$

and for $2 \leq k \leq m-1$:

$$\frac{\delta^2\Pi(d_k)}{\delta p_k \delta p_{k-1}} = \frac{2}{d_k-d_{k-1}} f\left(\frac{p_k-p_{k-1}}{d_k-d_{k-1}}\right) + \frac{p_k-c_k-p_{k-1}+c_{k-1}}{d_k-d_{k-1}} \frac{\delta f\left(\frac{p_k-p_{k-1}}{d_k-d_{k-1}}\right)}{\delta p_k}.$$

For $3 \leq k \leq m-1$ and $2 \leq l \leq k-1$ we also have $\delta^2\Pi(d_k)/\delta p_{k-l}^2 = 0$. Finally,

$$\begin{aligned}\frac{\delta^2\Pi(d_M)}{\delta p_m^2} &= -\frac{2}{d_m-d_{m-1}} f\left(\frac{p_m-p_{m-1}}{d_m-d_{m-1}}\right) - \frac{p_m-c_m-p_{m-1}+c_{m-1}}{d_m-d_{m-1}} \frac{\delta f\left(\frac{p_m-p_{m-1}}{d_m-d_{m-1}}\right)}{\delta p_m} \\ \frac{\delta^2\Pi(d_M)}{\delta p_m \delta p_{m-1}} &= \frac{2}{d_m-d_{m-1}} f\left(\frac{p_m-p_{m-1}}{d_m-d_{m-1}}\right) + \frac{p_m-c_m-p_{m-1}+c_{m-1}}{d_m-d_{m-1}} \frac{\delta f\left(\frac{p_m-p_{m-1}}{d_m-d_{m-1}}\right)}{\delta p_m},\end{aligned}$$

and for $2 \leq l \leq m-1$ we have $\delta^2\Pi(d_m)/\delta p_{m-l}^2 = 0$. In the above derivations we use the fact that $\delta f((p_2-p_1)/(d_2-d_1))/\delta p_1 = -\delta f((p_2-p_1)/(d_2-d_1))/\delta p_2$. Replacing the terms obtains the result. □

To show that p_T^* is a maximum for $\Pi(T)$, H should be negative definite at p_T^* (Güler, 2010). A matrix is negative definite if and only if all its leading principal minors of odd-numbered order are negative and those of even-numbered order are positive. The k-th order leading principal minor, D_k is the determinant of the k-th order principal sub-matrix formed by deleting the last $n-k$ rows and columns (Güler, 2010).

Lemma 9. *The leading principal minors of H, given in (19), are as following. For $2 \leq k \leq m-1$, k-th leading principal minor of H is $D_k = (-1)^k \sum_{l=1}^{k+1} \prod_{j=1}^{k+1} b_j/b_l$. Also, m-th leading principal minor of H is $D_m = (-1)^m \prod_{j=1}^{m} b_j$.*

Proof. For $k=2$ we have $D_2 = (-b_1-b_2)(-b_2-b_3) - b_2^2 = b_2b_3 - b_1b_3 + b_1b_2$. Thus, the formula works for 2. We use transfinite induction and suppose that the formula works for all $l \leq k$ with $2 \leq k < m-1$. In order to do so, first note that we have

$$D_{k+1} = \begin{vmatrix} -b_1-b_2 & b_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k-1} & -b_{k-1}-b_k & -b_k & 0 \\ 0 & 0 & \dots & 0 & b_k & -b_k-b_{k+1} & b_{k+1} \\ 0 & 0 & \dots & 0 & 0 & b_{k+1} & -b_{k+1}-b_{k+2} \end{vmatrix}$$

According to Laplace expansion, the determinant of a matrix can be obtained from its minors. Thus

$$D_{k+1} = (-1)^{2k+1} b_{k+1} D + (-1)^{2k+2}(-b_{k+1} - b_{k+2}) D_k = -b_{k+1} D - (b_{k+1} + b_{k+2}) D_k$$

where

$$D = \begin{vmatrix} -b_1 - b_2 & b_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k-1} & -b_{k-1} - b_k & 0 \\ 0 & 0 & \dots & 0 & b_k & b_{k+1} \end{vmatrix}$$

The matrix whose determinant is D has a column with only one non-zero entry. Thus we have $D = (-1)^{2k} b_{k+1} D_{k-1} = b_{k+1} D_{k-1}$. Therefore, we have

$$D_{k+1} = -b_{k+1}^2 D_{k-1} - (b_{k+1} + b_{k+2}) D_k \tag{20}$$

Based on the assumption of induction it holds that $D_{k-1} = (-1)^{k-1} \sum_{l=1}^{k} \prod_{j=1}^{k} b_j / b_l$, and $D_k = (-1)^k \sum_{l=1}^{k+1} \prod_{j=1}^{k+1} b_j / b_l$. By substituting the terms in (20) we get:

$$\begin{aligned} D_{k+1} &= -b_{k+1}^2 (-1)^{k-1} \sum_{l=1}^{k} \frac{\prod_{j=1}^{k} b_j}{b_l} - (b_{k+1} + b_{k+2})(-1)^k \sum_{l=1}^{k+1} \frac{\prod_{j=1}^{k+1} b_j}{b_l} \\ &= (-1)^{k-1} \left[-b_{k+1}^2 \sum_{l=1}^{k} \frac{\prod_{j=1}^{k} b_j}{b_l} + b_{k+1} \sum_{l=1}^{k+1} \frac{\prod_{j=1}^{k+1} b_j}{b_l} + \sum_{l=1}^{k+1} \frac{\prod_{j=1}^{k+2} b_j}{b_l} \right] \\ &= (-1)^{k-1} \left[-b_{k+1}^2 \sum_{l=1}^{k} \frac{\prod_{j=1}^{k} b_j}{b_l} + b_{k+1}^2 \sum_{l=1}^{k} \frac{\prod_{j=1}^{k} b_j}{b_l} + b_{k+2} \frac{\prod_{j=1}^{k+1} b_j}{b_{k+2}} + \sum_{l=1}^{k+1} \frac{\prod_{j=1}^{k+2} b_j}{b_l} \right] \\ &= (-1)^{k-1} \left[\frac{\prod_{j=1}^{k+2} b_j}{b_{k+2}} + \sum_{l=1}^{k+1} \frac{\prod_{j=1}^{k+2} b_j}{b_l} \right] = (-1)^{k+1} \sum_{l=1}^{k+2} \frac{\prod_{j=1}^{k+2} b_j}{b_l} \end{aligned}$$

This proves the induction premise in case $k \le m - 1$.

To obtain the formula for case $k = m$, first note that we have

$$D_m = \begin{vmatrix} -b_1 - b_2 & b_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m-2} & -b_{m-2} - b_{m-1} & -b_{m-1} & 0 \\ 0 & 0 & \dots & 0 & b_{m-1} & -b_{m-1} - b_m & b_m \\ 0 & 0 & \dots & 0 & 0 & b_m & -b_m \end{vmatrix}$$

According to Laplace expansion, the determinant of a matrix can be obtained from its minors. Thus

$$D_m = (-1)^{2m-1} b_m D + (-1)^{2m}(-b_m) D_{m-1} = -b_m D - b_m D_{m-1}$$

Similar to the previous case it can be observed that $D = b_m D_{m-2}$ and consequently

$$D_m = -b_m^2 D_{m-2} - b_m D_{m-1} \tag{21}$$

Using the assumption of induction we have

$$
\begin{aligned}
D_{k+1} &= -b_m^2(-1)^{m-2}\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} - b_m(-1)^{m-1}\sum_{l=1}^{m}\frac{\prod_{j=1}^{m} b_j}{b_l} \\
&= (-1)^{m-2}\left[-b_m^2\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} + b_m^2\sum_{l=1}^{m-1}\frac{\prod_{j=1}^{m-1} b_j}{b_l} + b_m\frac{\prod_{j=1}^{m} b_j}{b_m}\right] = (-1)^m\prod_{j=1}^{m} b_j.
\end{aligned}
$$

This proves the induction premise in case $k = m$. □

We are now ready to present the result which establishes the nature of the Hessian matrix H at the critical point of $\Pi(T)$ upon existence.

Lemma 10. *Let p_T^* be the vector of prices obtained via implicit equations in (18). The Hessian matrix H associated with $\Pi(T)$ is negative definite at p_T^*.*

Proof. Considering that equations (18) hold, it can be inferred from Lemma 1 that $b_k > 0$ for every $1 \leq k \leq m$. Given the formulas of leading principal minors of H it can be observed that at p_T^*, for odd k we have $D_k < 0$ and for even k we have $D_k > 0$. Therefore, $\Pi(p_T, d_T)$ is negative definite at p_T^*. □

Subsequently, we conclude that p_T^* upon existence is the maximum for $\Pi(T)$. We are now ready to state the main result of this section regarding the optimal prices of uptime-guaranteed contract menus.

Theorem 2. *Suppose that an admissible guaranteed-uptime level menu d_T is given. The system of equations in (18) obtains the prices which result in the highest expected profit for the supplier among all essential and profitable contract menus with d_T.*

Theorem 2 obtains the optimal prices of contract menus with admissible vectors of guaranteed uptime levels. It establishes the fact that for every vector of admissible guaranteed uptime levels, a unique vector of optimal prices exists. On the other hand, Lemma 5 clarifies that admissibility of a guaranteed uptime levels is a necessary condition for having essential and profitable contract menus. Hence, we can make the following conclusion regarding the conditions for existence of essential and profitable contract menus.

Corollary 3. *Let d_T be a vector of guaranteed uptime levels. A unique essential and profitable contract menu with the highest expected profit for the service provider exists if and only if d_T is admissible.*

Although we have solved the problem of optimal pricing of contract menus with given vector of guaranteed uptime levels, we still need to comment on the dilemma of finding appropriate uptime levels. Unlike the case of single price contracts, finding the best uptime levels for contract menus is rather cumbersome. First of all, it can be easily verified that the service provider's expected profit is non-decreasing on the number of contracts in a contract menu. Secondly, with a fixed number of contracts in a contract menu, the problem of finding the best choice of guaranteed uptime levels is combinatorial in nature. Nevertheless, if the

d	c	p^*
0.84	4,800	22,400
0.85	7,500	28,750
0.86	10,800	35,400
0.87	14,700	42,350
0.88	19,200	49,600
0.89	24,300	57,150
0.90	30,000	65,000
0.91	36,300	73,150
0.92	43,200	81,600
0.93	50,700	90,350
0.94	58,800	99,400
0.95	67,500	108,750
0.96	76,800	118,400

Table 1: Costs and optimal prices

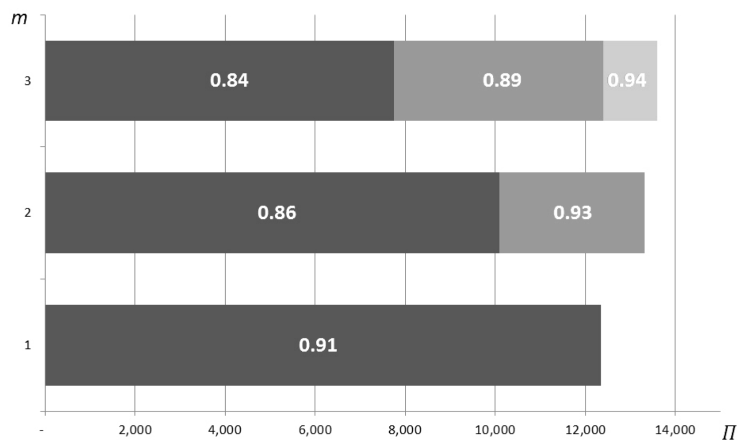

Figure 4: Expected profits of contract menus

number of contracts is small enough, which is usually the case in practice, one can enumerate all possible combination of contracts and find the most profitable combination. We finish the paper with an example which calculates the optimal contract menus of maximum size of three.

Example 1. *Consider that the customer's valuation is distributed uniformly between 0 and 1,000,000 units of money, that is, increasing the uptime level of the equipment over the period of one year can result in a revenue anywhere in this range for the customer. Without a contract the expected uptime of the equipment is %80. We normalize the preventive maintenance costs to zero. The service provider incur the cost of* $300000(d-d_0)^2$ *to offer a contract which has the guaranteed uptime level of d. The costs and optimal prices associated with offering a single contract with different guaranteed uptime levels are presented in Table 1.*

The optimal choice of guaranteed uptime level for a single contract is 0.91*. Accordingly, the optimal single contract has the expected profit of* $12,345$ *for the service provider. If the service provider wishes to offer a menu of contracts of size two, our numerical calculations indicate that the optimal contracts are* $t_1 = (50700, 0.86)$ *and* $t_2 = (90350, 0.93)$ *with expected profit of* $13,322$*. With a contract menu of size three, the service provider can capture the expected profit of* $13,597$ *when offering the optimal contracts* $t_1 = (22400, 0.84)$*,* $t_2 = (57150, 0.89)$*, and* $t_3 = (99400, 0.94)$*. Figure 4 depicts the expected profits of these contract menus. As seen in the figure, by increasing the number of contracts in the menu the spectrum of optimal guaranteed uptime levels widens.* △

As illustrated in the above Example, offering additional contracts will never decrease the expected profit of the service provider. Also, by including more contracts in a contract menu, the service provider can cater to a customer with low valuation while offering contracts of higher guaranteed service levels to extract the most of a customer with high valuation who is willing to pay more to reduce the probability of its equipment downtime.

6 Concluding Remarks

In this paper we solved the revenue optimization problem for a service provider who offers uptime-guarantee maintenance contracts. The novelty of our analysis is to take into consideration the fact that the service provider is unaware of actual valuation of the customer. By taking a value-based approach to contract optimization problem, we elaborated on ways to allow the service provider to leverage the probability of contract being purchased and the price of the contract. To do this, we abstracted the costs associated with offering the contracts and the average uptime of equipment relying solely on corrective maintenance activities. This black-box approach to costs enabled us to present closed-form formulas for optimal prices of contract menus comprising arbitrary number of individual contracts. Our results hold for the large family of IGFR distribution functions.

As the service provider increases the number of options for the equipment owner, it can expect higher total profit. The reason is that offering multiple contracts in a contract menu allows the service provider to extract the most out of its services by charging proportionally higher amounts for contracts with higher uptime levels to take advantage of the possible high-valuation customer. We provide direct formulas to calculate optimal prices given vectors of guaranteed uptime levels and suggested a brute force approach to find the optimal vectors of guaranteed uptime levels. Since managing contract menus which include large number of contracts is complicated and expensive in practice, the latter can be done successfully for contract menus with small number of contracts. In dealing with the combinatorial problem of finding optimal vectors of guaranteed uptime levels, smart algorithms are needed. This is an interesting next step which we leave for future research.

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