Beta
Research School for Operations Management and Logistics

Optimizing usage and maintenance decisions for *k*-out-of-*n* systems of moving assets

A.S. Eruguz, T. Tan, G.J. van Houtum

Beta Working Paper series 525

BETA publicatie	WP 525 (working paper)
ISBN	
ISSN	
NUR	
Eindhoven	March 2017

Optimizing usage and maintenance decisions for *k*-out-of-*n* systems of moving assets

Ayse Sena Eruguz, Tarkan Tan, Geert-Jan van Houtum

School of Industrial Engineering, Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

March 10, 2017

Abstract: We consider an integrated usage and maintenance optimization problem for a *k*-out-of-*n* system pertaining to a moving asset. The *k*-out-of-*n* systems are commonly utilized in practice to increase availability, where *n* denotes the total number of parallel and identical units and *k* the number of units required to be active for a functional system. Moving assets such as aircrafts, ships, and submarines are subject to different operating modes. Operating modes can dictate not only the number of system units that are needed to be active, but also where the moving asset physically is, and under which environmental conditions it operates. We use the intrinsic age concept to model the degradation process. The intrinsic age is analogous to an intrinsic clock which ticks on a different pace in different operating modes. In our problem setting, the number of active units, degradation rates of active and standby units, maintenance costs, and type of economic dependencies are functions of operating modes. In each operating mode, the decision maker should decide on the set of units to activate (usage decision) and the set of units to maintain (maintenance decision). Since the degradation rate differs for active and standby units, the units to be maintained depend on the units that have been activated, and vice versa. In order to minimize maintenance costs, usage and maintenance decisions should be jointly optimized. We formulate this problem as a Markov decision process and provide some structural properties of the optimal policy. Moreover, we assess the performance of usage policies that are commonly implemented for maritime systems. We show that the cost increase resulting from these policies is up to 27% for realistic settings. Our numerical experiments demonstrate the cases in which joint usage and maintenance optimization is more valuable.

Keywords: *k*-out-of-*n* system, maintenance, usage, maritime, moving assets

1. Introduction

For moving assets such as aircrafts, ships, and submarines, unexpected downtimes can be very costly. In addition, failures can affect health, safety, and environment. Redundancy, i.e., having a number of identical units in parallel, is the most obvious way to prevent downtime. The *k*-out-of-*n* systems are common in practice, where *n* states the number of parallel units and *k* the number of units that needs to be functioning.

The operations of moving assets consist of different operating modes. In each operating mode, degradation parameters and operational requirements are different. For example, the operating modes of aircrafts consist of standby, taxiing, take-off, cruising, and landing (Özekici and Soyer, 2004). The operating modes of a Navy ship reflects its physical location, environment, and mission type, e.g., harbor, transit, training, anti-submarine warfare, and surveillance (Tinga and Janssen, 2013). In each operating mode, the number of engines that is required to be active is different. For the above-given aircraft example, one engine is required during the taxi phase, but both engines are necessary during the take-off phase (Xing et al., 2012). Similarly, a ship can use several propulsion engines while sailing in high speed, but a single propulsion engine is sufficient for slow speed operations and during harbor visits. Other systems with similar characteristics include the energy generation systems and refrigeration plants on board of the frigates (Tinga and Janssen, 2013), aerospace computing systems (Bricker, 1973; Xing et al., 2012), and airborne weapon systems (Winokur and Goldstein, 1969). For the above-given examples, the sequence and the duration of operating modes can be modelled as random variables (see, e.g., Alam and Al-Saggaf, 1986; Çekyay and Özekici, 2015).

Moving assets operate at remote locations and are exposed to different stress levels and environmental conditions in different operating modes (Xing et al., 2012; Levitin et al., 2016). Consequently, the rate at which the system units degrade varies. Moreover, the units that are in standby mode usually degrade with a lower rate than the units that are active. In this paper, we use the *intrinsic age concept* to model this degradation process (see, e.g., Çinlar and Özekici, 1987; Özekici, 1995; Çekyay and Özekici, 2015). The intrinsic age is analogous to an intrinsic clock which ticks on a different pace in different operating modes for active and standby units. In practice, a unit can be defined as failed if its overhaul cannot be delayed any further. This corresponds to a soft failure in case of which the unit should be switched-off and cannot be utilized until being overhauled. For complex units such as maritime or aircraft engines, conventional practice has been to perform an overhaul for which the timing is determined by running hours. Intrinsic age better reflects the influence of the stochastic mission process and usage decisions on degradation.

In this paper, we consider that the intrinsic age of the units is *observable*. This enables condition-based usage and maintenance decisions, i.e., deactivating a unit or executing maintenance at the moment that its intrinsic age exceeds a certain threshold value. The condition of system units can be monitored continuously or periodically. Currently, the applicability of continuous monitoring is limited for many moving assets. Continuous monitoring requires an investment in online sensor technologies. Only 2% of the classed world maritime fleet currently employs such techniques (Tinsley, 2016). Most commonly, physical inspections are performed by the personnel on board or at the home base (e.g., harbor, dock, or hangar).

Because of the existence of economic dependencies in a k-out-of-n system, the optimal maintenance decision on system level is not simply the combination of the optimal maintenance decisions on unit level (Dekker et al., 1997; Nicolai and Dekker, 2008). In some cases, it might be beneficial to maintain several units simultaneously to benefit from economies of scale and to reduce the setup costs (positive economic dependence). For other cases, maintaining several units simultaneously is more expensive than maintaining them individually (negative economic dependence). This is due to the resulting peak in manpower needs, combined risk of human error, the cost of downtime, and legal/safety requirements prohibiting joint maintenance. Economic dependencies considered in our model have a general structure. For moving assets, we often observe positive economic dependence when the moving asset is at the home base and negative economic dependence during missions at remote locations. In order to make the best use of positive economic dependence and exploit the lifetime of the units, the intrinsic age of several units has to be sufficiently high just before maintaining them. On the other hand, if the intrinsic ages of several units are high, the reliability of the whole system decreases during missions. In this case, maintenance of these units may become inevitable in a mission state where the value of k is high and economic dependence is negative. The value of k is dictated by the operating mode. The decision maker should decide which units to activate among the functional units. Usage decisions influence future maintenance decisions (i.e., the units to be maintained) and maintenance decisions influence future usage decisions (i.e., the units to be activated). In order to minimize maintenance costs, usage and maintenance decisions should be jointly optimized.

Our work is motivated by our collaboration with two maritime asset owners, Fugro and the Royal Netherlands Navy (RNLN). Fugro provides geotechnical, survey, subsea, and geoscience services. A major part of Fugro's turnover stems from the research of the seafloor, for which the so-called survey vessels are employed. The propulsion system of the survey vessels contains three diesel engines. The number of engines to activate varies during operations. Maintenance of these engines is outsourced to the Original Equipment Manufacturer (OEM) and maintaining several engines simultaneously is not preferable because of negative economic dependencies. The currently implemented usage policy is to activate the most deteriorated unit(s) first (unbalanced usage policy). Maintenance services are requested from the OEM when the most deteriorated unit reaches its soft failure threshold. The OEM can maintain the diesel engines on board. However, setup and maintenance costs depend on the physical location of the vessel. There exist similar system configurations for the navy frigates of the RNLN. The navy frigates have four diesel generators that satisfy varying energy requirements of the frigate. The generators are always operated in fixed pairs (see Tinga and Janssen, 2013). It is preferable to maintain these units simultaneously in the home harbor due to positive economic dependence. However, maintenance during missions has to be avoided because of safety concerns and negative economic dependence in mission states. The RNLN utilizes the two pairs of diesel generators equally during missions (balanced usage policy) and maintain them altogether in the home harbor when they are sufficiently deteriorated. In this paper, we assess the performance of these commonly implemented policies. Our numerical experiments show that the integration of usage and maintenance decisions has a great value in these real-life settings.

Our main contributions to the literature are as follows. To the best of our knowledge, the above-presented k-out-of-n system setting has not been considered in the maintenance optimization literature despite its practical relevance. We explicitly model the relation between the system's usage and deterioration in different operating modes and optimize usage and maintenance decisions in an integrated way. We formulate this optimization problem as a Markov decision process in which the objective is to minimize the expected total discounted maintenance cost. We provide structural properties of the optimal policy and give insights into the performance of the optimal policy compared to simple policies that are commonly used in practice.

The paper is organized as follows. Section 2 presents the related literature. Section 3 formulates the problem as a Markov decision process. Section 4 characterizes the structure of the optimal usage and maintenance policy. Section 5 provides numerical experiments and an illustrative example, providing insights about the performance of the optimal policy. Section 6 draws some conclusions and suggests potential future research directions.

2. Literature Review

Our definition of moving assets is analogous to the so-called *phased-mission systems* which are subject to multiple phases of operation. The environment under which these systems operate can change from phase to phase as well as their failure properties. There is a vast literature on reliability analysis of phased-mission systems (e.g., Esary and Ziehms, 1975; Kim and Park, 1994; Kharoufeh et al., 2010). However, only a few papers include the redundancy aspect. The settings considered in Xing et al. (2012) and Rokseth and Utne (2015) are very similar to that of our paper. Xing et al. (2012) investigate the exact reliability evaluation of k-out-of-n systems with identical units subject to phased-mission requirements. The value of k and failure time distributions of system units change from phase to phase. The proposed method is demonstrated on examples inspired from multi-processor computer systems of aerospace assets. Rokseth and Utne (2015) perform risk assessment focusing on diesel propulsion engines in dynamic positioning systems of maritime and offshore vessels. The risk assessment of Rokseth and Utne (2015) considers potential future failures.

Recently, a new type of optimization problem, called as the *standby element sequencing problem*, has been investigated for 1-out-of-n systems with non-identical units that are subject to phased-mission requirements (see, e.g., Levitin et al., 2014; Dai et al., 2016; Levitin et al., 2016). In this problem, the objective is to select the activation sequence of units so as to minimize the expected mission cost of the system while providing a certain level of system reliability. It is assumed that during different mission phases the time-to-failure distributions of units are different. In our paper, usage decisions consist of activation and deactivation of units and it has similarities with the above-mentioned sequencing decision. However, we focus on k-out-of-n systems with identical units. Active and standby units degrade with different rates in different operating modes (i.e., mission phases) and the number units to be activated (i.e., the value of k) depends on the operating mode. At inspection epochs, the system condition (i.e., the intrinsic age of the units) is inspected and usage and maintenance decisions are updated given the current condition and operating mode.

Usage and maintenance decisions that we consider are condition-based decisions. In the last two decades, several condition-based maintenance models have been developed for phased-mission systems (see, e.g., Özekici, 1995; Çekyay and Özekici, 2012; Ulukus et al. 2012; Xiang et al., 2012; Flory et al., 2015). Our work is closely related to that of Özekici (1995) and Ulukus et al. (2012). Similar to our work, the authors consider the intrinsic age concept for degradation. The intrinsic age is a continuous random variable, and its pace is modulated by the mission process. The structure of optimal maintenance decisions is characterized for finite state Markov processes. However, these papers neither consider redundancy nor varying operational requirements in different mission phases. Therefore, usage decisions are not needed in their problem setting.

For multi-unit systems, the optimal maintenance decision on system level depends on economic dependencies. A number of maintenance models have been proposed for multi-unit systems, but most of them consider positive economic dependencies (see e.g., Wildeman et al., 1997; Bouvard et al., 2011; Zhang et al., 2013; Çekyay and Özekici, 2015; Zhu et al., 2015). Vu et al. (2014) considers both negative and positive economic dependence in complex multi-unit systems having a mixture of series and parallel connections. Huynh et al. (2015) and Olde Keizer et al. (2016) propose condition-based maintenance policies for *k*-out-of-*n* systems with positive economic dependence. Sheu and Kuo (1994) and Pham and Wang (2000) consider both positive and negative economic dependencies for *k*-out-of-*n* systems. To the best of our knowledge, in none of the existing papers the structure of economic dependencies is operating mode dependent.

3. Markov decision process formulation

In this section, we formulate the optimization problem of usage and maintenance decisions in a *k*-out-of-*n* system as a Markov decision process (MDP). Since many moving assets have very long service times, we develop an infinite horizon model. Our objective is to minimize the expected total discounted cost of maintenance over an infinite horizon.

Let I be a finite non-empty set representing different operating modes of a moving asset. Operating modes dictate the number of units that are required to be active (influencing possible usage decisions), where the moving asset physically can be (influencing cost parameters), and under which environmental conditions it operates (influencing degradation parameters). For many moving assets, the sequence and the duration of operating modes are more realistically modelled as random variables since there exists various uncertainties about the future operations and environments (see Alam and Al-Saggaf, 1986; Çekyay and Özekici, 2015). The operation process of moving assets can be well-presented by a Markov process (see Çekyay and Özekici, 2012; Eruguz et al., 2016). In this respect, we assume that the operation process evolves as a continuous-time Markov chain (CTMC) on discrete state space I. The sojourn time in each operating mode $i \in I$ is exponentially distributed with rate

μ_i. The probability that the system jumps to operating mode $j \in I$ when it leaves operating mode $i \in I$ is denoted by π_{ij}. Without loss of generality, we assume that $\pi_{ii} = 0$.

Inspections of the system reveal the current condition of the units in the *k*-out-of-*n* system and correspond to the decision epochs. We assume that the system is inspected and a decision regarding maintenance and usage is made when the operating mode changes. Additional inspection and decision epochs occur according to a Poisson process with rate $r - \mu_i$ in operating mode $i \in I$ where $r - \max\{\mu_i \mid i \in I\} \geq 0$. When the value of r is very high, inspections approach to continuous monitoring. On the other hand, when $r = \max\{\mu_i \mid i \in I\}$, inspections are dictated by the operation process. For $i, j \in I$, the operating mode transition probabilities between two consecutive inspection epochs can be computed by:

$$p_{ij} = \begin{cases} \pi_{ij}\mu_i / r & \text{for}\, i \neq j \\ 1 - \sum_{j \in I} \pi_{ij}\mu_i / r & \text{for}\, i = j. \end{cases}$$

Let $S = \{1, \dots, n\}$ be the finite set of identical units composing the *k*-out-of-*n* system (where $n \geq 2$). The number of units that are required to be active $k \in \{0, 1, \dots, n\}$ is a function of operating modes. The set of operating modes in which *k* units are required to be active is denoted by I_k where:

$$I = \bigcup_{k=0}^{n} I_k \, .$$

The *k*-out-of-*n* system is subject to different environmental conditions and stress levels in different operating modes. We model this by modulating the degradation rate by the operation process. The degradation of the units accumulates with linear and deterministic rates in different operating modes. The stochasticity in the degradation process stems from the operation process. Such degradation processes are commonly considered by practitioners from aviation and manufacturing industry (Özekici, 1995; Ulukus et al., 2012). They are commonly applied in reliability analysis and maintenance optimization (see, e.g., Kharoufeh, 2003; Kharoufeh and Cox, 2005; Kharoufeh et al., 2010; Xiang et al., 2012).

In the *k*-out-of-*n* system, the degradation rates differ based on usage, i.e., depending on whether a unit is active or on standby. The degradation rate in operating mode $i \in I$ is denoted by $\lambda_i^1 > 0$ for active units and $\lambda_i^0 \geq 0$ for standby units. The time required to activate a standby unit or deactivate an active unit is assumed to be negligible. Without loss of generality, we assume that $\lambda_i^1 \geq \lambda_i^0$.

The total degradation accumulated on unit $s \in S$ corresponds to its intrinsic age $d_s \in \Re_+$ where $\Re_+$ is the set of non-negative real numbers. The intrinsic age accumulates continuously and additively over time with the above-given degradation rates. A unit is functional if its intrinsic age is less than a certain *soft failure threshold F*. That is, if $d_s \geq F$, it is not safe to activate unit $s \in S$ and the unit becomes non-functional. Non-functional units have to be switched-off or maintained. The units that are switched-off do not degrade any further. For critical systems on board of the moving assets, inspection rate r is assumed to be large enough, making the likelihood of a hard failure between two inspection instants negligible.

The decision maker inspects the current operating mode $i \in I$ and the current system condition $\mathbf{d} = (d_s \mid s \in S)$ at each inspection instant and makes a decision. The action space consists of $2n$-dimensional action vectors $\mathbf{a} = (\mathbf{m}, \mathbf{u})$, where $\mathbf{m} = (m_s \mid s \in S)$ represents a binary vector of maintenance actions and $\mathbf{u} = (u_s \mid s \in S)$ represents a binary vector of usage actions. If unit $s \in S$ is to be maintained $m_s = 1$, otherwise, $m_s = 0$. Similarly, if unit $s \in S$ is to be activated $u_s = 1$, otherwise, $u_s = 0$.

We assume that maintenance actions are instantaneous and they are immediately followed by usage actions. For critical systems, the maintenance lead time is usually in the order of hours or days while time-to-failure is in the order of months or years. Therefore, it is justified to assume that the maintenance lead time is negligible. The downtime resulting from maintenance can be reflected in the cost parameters.

The condition of a unit after maintenance is assumed to be as good as new, i.e., $d_s = 0$ for all maintained units $s \in S$ after maintenance. Subsequently, degradation accumulates with rate λ_i^1 for active units and rate λ_i^0 for standby units in operating mode $i \in I$ until the next inspection epoch. For ease of exposition in the MDP, we swap the index of the units such that $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ after maintenance and usage actions. Hence, we define the set of condition vectors as $\Omega = \{(d_1, \ldots, d_n) \in \Re_+^n \mid d_1 \geq d_2 \geq \ldots \geq d_n \geq 0\}$ and the state space of the Markov decision process as $I \times \Omega$.

In operating mode $i \in I_k$, if the number of functional units is less than *k*, then maintenance should be performed immediately to attain at least *k* functional units. We assume that maintenance is possible in each operating mode. This is realistic for maritime assets since most of the maintenance activities can be performed on board. Let $\Omega_l \subset \Omega$ be the set of system condition vectors in which exactly $l \in \{0,1,\ldots,n\}$ units are functional. If $\mathbf{d} \in \Omega_l$, then:

$$d_1 \geq \ldots \geq d_{n-l} \geq F > d_{n-l+1} \geq \ldots \geq d_n \geq 0 \quad \text{for } l \in \{1,2,\ldots,n\}$$

$$d_1 \geq \ldots \geq d_{n-l} \geq d_{n-l+1} \geq \ldots \geq d_n \geq F \quad \text{for } l = 0.$$

The set of possible maintenance actions $M(i,\mathbf{d})$ consists of maintenance action vectors satisfying the following property:

$$\sum_{s\in S} m_s \geq \max\{k-l,0\} \qquad \text{for } i\in I_k, \mathbf{d}\in\Omega_l. \tag{1}$$

Let $\mathbf{e}_s$ be the n-dimensional unit vector with a 1 on the s-th position. Let Θ be an operator that orders the system units in descending order of intrinsic age. The effect of maintenance action $\mathbf{m}$ on system's condition vector $\mathbf{d}$ is modelled using operator $\mathrm{M}_{\mathbf{m}}$, which sets the intrinsic age of all maintained units to zero:

$$\mathrm{M}_{\mathbf{m}}\,\mathbf{d} = \Theta\left(\mathbf{d} - \sum_{s\in S} m_s d_s \mathbf{e}_s\right). \tag{2}$$

Each unit $s\in S$ can either be active ($u_s = 1$), in standby ($u_s = 0$ and $d_s < F$), or switched-off ($u_s = 0$ and $d_s \geq F$). The set of possible usage actions $U(i,\mathbf{d})$ consists of usage action vectors satisfying the following property:

$$\sum_{s\in S, d_s<F} u_s = k \qquad \text{for } i\in I_k, \mathbf{d}\in\Re_+^n. \tag{3}$$

Let $\mathrm{U}_{\mathbf{u},i,t}\,\mathbf{d}$ be the usage operator modelling the effect of the usage action $\mathbf{u}$ on condition vector $\mathbf{d}$, if the system resides in operating mode $i\in I$ during x time units:

$$\mathrm{U}_{\mathbf{u},i,x}\,\mathbf{d} = \Theta\left(\mathbf{d} + \sum_{s\in S, d_s<F} \lambda_i^{u_s} x \mathbf{e}_s\right). \tag{4}$$

Since the usage follows maintenance, the units that are maintained at a decision epoch can be activated immediately. Hence at a decision epoch, if the system's state is $(i,\mathbf{d})\in I\times\Omega$, a possible decision is defined as $\mathbf{a} = (\mathbf{m},\mathbf{u})$ where $\mathbf{m}\in M(i,\mathbf{d})$ and $\mathbf{u}\in U(i, \mathrm{M}_{\mathbf{m}}\,\mathbf{d})$.

In order to perform maintenance, maintenance engineers, equipment, and spare parts usually need to be transported to the moving asset, which results in high costs when the moving asset is outside the home base. As a consequence, maintenance costs depend on the operating mode in which maintenance is performed. The maintenance actions performed before or after crossing the soft failure threshold do not differ since the likelihood of a hard failure between two inspection instants is negligible. Therefore, we do not distinguish between corrective and preventive costs. Maintenance costs consist of setup costs and per-unit maintenance costs. Per-unit maintenance costs $C^{\mathrm{var}}(i) > 0$ include variable costs of man-hours and spare parts required for the maintenance of a single unit in operating mode $i\in I$. Setup costs include the fixed costs associated with transportation, maintenance setup, and the associated downtime. Setup costs $C^{\mathrm{set}}(i,\eta) \geq 0$ do not only depend on the operating mode $i \in I$ but also on the number of units to be maintained η, where:

$$\eta = \sum_{s \in S} m_s \ .$$

and $C^{\mathrm{set}}(i,0) = 0$. The total maintenance cost $C^{\mathrm{tot}}(i,\eta)$ is defined as:

$$C^{\mathrm{tot}}(i,\eta) = \eta C^{\mathrm{var}}(i) + C^{\mathrm{set}}(i,\eta).$$

If maintaining several units simultaneously is less expensive than maintaining them separately, i.e. when there is positive economic dependence while maintaining $\eta \in \{2,...,n\}$ units in operating mode $i \in I$, then $C^{\mathrm{tot}}(i,\eta)$ is subadditive in η, i.e.:

$$C^{\mathrm{tot}}(i,\eta) \leq C^{\mathrm{tot}}(i,\eta - s) + C^{\mathrm{tot}}(i,s) \text{ for all } s \in \{0,1,...,\eta\}.$$

On the other hand, if maintaining several units simultaneously is more expensive than maintaining them separately, i.e., when there is negative economic dependence while maintaining $\eta \in \{2,...,n\}$ units in operating mode $i \in I$, then $C^{\mathrm{tot}}(i,\eta)$ is superadditive in η, i.e.:

$$C^{\mathrm{tot}}(i,\eta) \geq C^{\mathrm{tot}}(i,\eta - s) + C^{\mathrm{tot}}(i,s) \text{ for all } s \in \{0,1,...,\eta\}.$$

Since the variable cost component in $C^{\mathrm{tot}}(i,\eta)$ (i.e., $\eta C^{\mathrm{var}}(i)$) is linearly increasing in η, the subadditivity or superadditivity of $C^{\mathrm{tot}}(i,\eta)$ stems from the setup costs $C^{\mathrm{set}}(i,\eta)$. Our model can handle general structures for $C^{\mathrm{tot}}(i,\eta)$, i.e., we do not require that $C^{\mathrm{tot}}(i,\eta)$ is subadditive or superadditive.

We assume a continuous discount rate $\alpha > 0$ so that any cost incurred at some future time x is discounted by a factor $e^{-\alpha x}$. Let $V(i,\mathbf{d})$ be the value function representing minimum expected total discounted cost using the optimal policy, if the current state is $(i,\mathbf{d}) \in I \times \Omega$:

$$V(i,\mathbf{d}) = \min_{(\mathbf{m},\mathbf{u}) | \mathbf{m} \in M(i,\mathbf{d}), \mathbf{u} \in U(i, \mathrm{M}_{\mathbf{m}} \mathbf{d})} \left\{ C^{\mathrm{tot}}(i, \sum_{s \in S} m_s) + \Gamma_{\mathbf{m},\mathbf{u}} V(i,\mathbf{d}) \right\} \tag{5}$$

where $\Gamma_{\mathbf{m},\mathbf{u}}$ is an operator that models action $(\mathbf{m},\mathbf{u})$ with $\mathbf{m} \in M(i,\mathbf{d})$ and $\mathbf{u} \in U(i, \mathrm{M}_{\mathbf{m}} \mathbf{d})$ as follows:

$$\Gamma_{\mathbf{m},\mathbf{u}} V(i,\mathbf{d}) = \sum_{j \in I} p_{ij} \int_0^{\infty} r e^{-(r+\alpha)x} V\left(j, \mathrm{U}_{\mathbf{u},i,x} \mathrm{M}_{\mathbf{m}} \mathbf{d}\right) dx \tag{6}$$

Let $\mathbb{N}_0$ be the set of non-negative integers and $V_t(i,\mathbf{d})$ be the minimum expected total discounted cost when there are $t \in \mathbb{N}_0$ inspections left starting in operating mode $i \in I$ and condition vector $\mathbf{d} \in \Omega$. We define the *t*-stage problem for $t \in \mathbb{N}_0 \backslash \{0\}$ as follows:

$$V_t(i,\mathbf{d}) = \min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\mathbf{d}),\mathbf{u}\in U(i,\mathrm{M}_\mathbf{m}\mathbf{d})} \left\{ C^{\mathrm{tot}}(i, \sum_{s\in S} m_s) + \Gamma_{\mathbf{m},\mathbf{u}} V_{t-1}(i,\mathbf{d}) \right\} \tag{7}$$

Without loss of generality we can select $V_0(i,\mathbf{d})$ such that:

$$V_0(i,\mathbf{d}) = C^{\mathrm{tot}}\left(i, \max\{k-l, 0\}\right) \text{ for } i \in I_k, \mathbf{d} \in \Omega_l, \text{ and } k,l \in \{0,1,..,n\}. \tag{8}$$

Our model satisfies the conditions of Corollary 9.17.1 in Bertsekas and Shreve (2004, p. 235), which establishes the existence of an optimal deterministic stationary policy and the convergence of the value iteration algorithm to the optimal value. When solving problem (7) numerically, we discretize the continuous component of the state space (see Appendix I).

4. Optimal Policy

In this section we analyze the structure of the optimal usage and maintenance policy. Theorem 1 gives the structure of the optimal maintenance policy, which is valid regardless of the optimal usage policy. Theorem 2 states that the optimal usage policy has a specific structure if the optimal maintenance policy satisfies certain conditions. These structural properties stem from monotonicity results related to the system's condition. The proofs of the theorems are given in Appendix II and III, respectively.

Let $\mathbf{m}_s$ be the maintenance action vector having the value of 1 on positions 1,…,*s* and the value of 0 on positions $s+1$, …, *n*.

Theorem 1: The structure of the optimal maintenance policy satisfies the following:

(a) In state $(i,\mathbf{d}) \in I_k \times \Omega_l$ with $k,l \in \{0,1,\ldots,n\}$, there exists an $s \in \{\max\{0, k-l\},\ldots,n\}$ such that an optimal maintenance policy $\mathbf{m}^*(i,\mathbf{d})$ is $\mathbf{m}_s$.

(b) If $\mathbf{m}^*(i,\mathbf{d}) = \mathbf{m}_s$, then $\mathbf{m}^*(i, \Theta(\mathbf{d} + \varepsilon_{s'}\mathbf{e}_{s'})) = \mathbf{m}_s$ for any $\varepsilon_{s'} \in \Re^+$ and s' satisfying $1 \le s' \le s \le n$.

Theorem 1(a) states that the units that are to be maintained are the ones with the highest intrinsic age. Therefore, at each decision epoch, the number of units to be maintained is a sufficient indicator of the optimal maintenance decision. Theorem 1(b) implies the following. Assume that $\mathbf{m}_s$ is optimal in state $(i,\mathbf{d}) \in I \times \Omega$. Then $\mathbf{m}_s$ remains optimal in operating mode $i \in I$, if the conditions of the units that are to be maintained are worse and the conditions of the units that are not to be maintained are the same as the ones in $\mathbf{d} \in \Omega$.

From a practical standpoint, these properties show the optimality of an intuitive approach: maintaining the most deteriorated units first. They are also useful to reduce the computational effort in the solution algorithm. The action space can be reduced significantly by eliminating the maintenance decisions that do not satisfy these properties.

Let $\mathbf{u}_{s_1,s_2}$ be the usage vector for $s_1, s_2 \in S$ and $s_1 \leq s_2$ having the value of 1 on positions $s_1,...,s_2$ and the value of 0 on the remaining positions.

Theorem 2: If either $\mathbf{m}_0$ or $\mathbf{m}_n$ is optimal in each state $(i,\mathbf{d}) \in I \times \Omega$ for all t-stage problems where $t \in \mathbb{N}_0 \backslash \{0\}$, then the associated optimal usage decision is $\mathbf{u}_{n-k+1,n}$ in each operating mode $i \in I_k$ with $k \in \{1,...,n\}$ for all t-stage problems and also for the infinite horizon problem.

The maintenance policy given as a condition in Theorem 2 corresponds to an *all-or-nothing policy*, i.e., either maintain all units or none of them. Although this maintenance policy is restrictive, it is commonly applied for systems that are maintained by major overhauls (e.g., navy frigates). The usage policy described in Theorem 2 is equivalent to activate the least deteriorated k units in each operating mode $i \in I_k$. This tends to balance the intrinsic age of all units. We refer to this usage policy as *balanced usage policy* (BUP). When this usage policy is combined with an all-or-nothing maintenance policy, the objective is to equally use all of the units and maintain them simultaneously when they are sufficiently deteriorated. We note that for the optimality of an all-or-nothing maintenance policy, positive economic dependence while maintaining n units is necessary in all operating modes (see Remark 1 in Appendix IV). However, the latter is not a sufficient condition (see Table 4). Even when there is positive economic dependence while maintaining n units, maintaining a few units might be cost-effective in operating modes where maintenance is expensive. As such facultative maintenance tasks can be postponed to the operating modes where maintenance is cheap.

The counterpart of BUP is the *unbalanced usage policy* (UUP) under which the most deteriorated (yet functional) k units are activated in each operating mode $i \in I_k$. This usage policy intends to increase the intrinsic age difference between the most deteriorated (the oldest) and the least deteriorated (the youngest) functional units. As such, UUP exploits the benefits of redundancy within the system. We note that the optimality of UUP cannot be guaranteed as a counterpart of Theorem 2, i.e., under the optimality of a certain maintenance policy (see Remark 2 in Appendix IV). In maritime applications, we observe that UUP is usually preferred when there is negative economic dependence in all operating modes. However, negative economic dependence is neither necessary nor sufficient condition for the optimality of UUP. Indeed, UPP can be optimal even when there is positive economic dependence (see Remark 3 in Appendix IV). In Section 5, we assess the performance of BUP and UPP for realistic problem settings.

5. Numerical experiments

In maritime applications, we observe that operators apply either BUP or UUP mainly depending on the economic dependencies that are encountered. As a rule of thumb, BUP is adopted in case of positive economic dependence since it enables to exploit the economies of scale to the fullest extent. On the other hand, UUP exploits the redundancy in the system by increasing the intrinsic age difference between the oldest and the youngest units. In practice, UUP is favored in case of negative economic dependence. In a maritime application, there is positive economic dependence when the ship is docked. However, there is often negative economic dependence in mission states because of safety requirements, long downtime costs resulting from joint maintenance, and limited resource capacity during missions. Such a structure of economic dependencies makes the optimal usage policy non-obvious for practitioners.

In this section, we execute numerical experiments in order to test the overall performance of BUP and UUP. Maintenance decisions are optimized for both benchmark policies. In Section 5.1, we describe our test bed. The results of our comparative analysis are discussed in Section 5.2. In Section 5.3, we illustrate the optimal usage and maintenance policy on a problem instance.

5.1. Setup

We are motivated by maritime *k*-out-of-*n* systems such as the engines of a propulsion system, generator sets of an energy generation system, or water chillers of a refrigeration plant. These units are associated with critical functions, i.e., propulsion, energy generation, and cooling, respectively. We consider systems containing 2 or 3 units. We note that our test bed reflects our observations from real-life systems of both commercial (survey vessels of Fugro with 3 diesel propulsion engines) and defense equipment (frigates of the RNLN with 2 pairs of diesel generators and 3 water chillers).

We define 4 operating modes $I \equiv \{0,1,2,3\}$. Operating mode $i=0$ represents a state in which the ship is in dock (home harbor), none of the units are required to be active. Operating mode $i=1$ represents a state in which the ship is in a (foreign) harbor during which minimum power, i.e. one functional unit is sufficient. For a system consisting of 2 units, operating modes $i=2$ and $i=3$ are transit/mission states, in which 1 and 2 units are required to be active, respectively. For a system consisting of 3 units, operating modes $i=2$ and $i=3$ are transit/mission states, in which 2 and 3 units are required to be active, respectively. Dock visits are not very frequent. They are followed by equally probable transit/mission states. Transit/mission states can be followed by a harbor visit or another transit/mission state with the same probability. For such an operation process, the operating mode transition probability matrix is defined as in Table 1.

Three alternatives are considered regarding the operating mode transition rates representing constant, increasing, and decreasing transition rates as a function of $i \in I$. The different alternatives for operating mode transition rates are computed using transition coefficients δ_i given in Table 2. For each alternative, we set $\mu_0 = 6$ per year and apply $\mu_i = \delta_i \mu_0$ for each $i = \{1,2,3\}$. Similarly, three alternatives are considered for the degradation rates of active units, covering the cases in which the degradation is constant, increasing, and decreasing as a function of $i \in I$. For each alternative, we set $\lambda_0^1 = 0$ and $\lambda_i^1 = \delta_i$ for all $i = \{1,2,3\}$. Three alternatives are considered for the degradation rate of standby units such that $\lambda_i^0 = 0$ (none), $\lambda_i^0 = 0.5 \times \lambda_i^1$ (medium), and $\lambda_i^0 = 0.8 \times \lambda_i^1$ (high) for each $i \in I$.

Table 1: Operating mode transition probability matrix

π_{ij}	j = 0	j = 1	j = 2	j = 3
i = 0	0.000	0.000	0.500	0.500
i = 1	0.010	0.000	0.495	0.495
i = 2	0.010	0.495	0.000	0.495
i = 3	0.010	0.000	0.495	0.495

Table 2: Transition and degradation coefficients for each alternative

Transition coefficients δ_i	i = 1	i = 2	i = 3
Constant	1.00	1.00	1.00
Increasing	0.50	1.00	1.50
Decreasing	1.50	1.00	0.50

We note that by the definition of operating modes, maintenance costs are non-decreasing as a function of $i \in I$. We consider increasing and constant maintenance cost alternatives. Typically, maintenance costs are higher in transit/mission states than in the harbor. The lowest maintenance cost is incurred in dock. In addition, maintenance costs might be higher in $i = 3$ than in $i = 2$, considering the physical location of the moving asset or the resulting downtime when operational requirements in terms of the number of active units is high. Under the latter situation, maintenance costs are increasing as a function of $i \in I$. On the other hand, for some systems, there is a possibility to have spare part stocks and trained maintenance personnel on board at all times, which would make maintenance costs constant over operating modes. The cost coefficients considered for increasing and constant maintenance cost alternatives are as given in Table 3. Using cost coefficient κ_i in operating mode $i \in I$, we compute the fixed cost of maintaining a single unit by:

$C^{\text{fix}}(i,1) = \kappa_i C^{\text{var}}(i)$ where $C^{\text{var}}(i) =$ €1,000 for all $i \in I$.

We consider positive, negative, and mixed economic dependencies. For each operation mode $i \in I$, setup costs are given by:

$$C^{\text{set}}(i,\eta) = C^{\text{set}}(i,1)\sqrt{\eta} \quad \text{for } \eta \in \{2,3\} \tag{9}$$

which is a subadditive function in the number of units to be maintained $\eta = \{2,3\}$, representing positive economic dependence in all operating modes. Superadditive function:

$$C^{\mathrm{set}}(i,\eta) = C^{\mathrm{set}}(i,1)\eta^2 \text{ for } \eta \in \{2,3\} \tag{10}$$

is used to represent negative economic dependence in all operating modes. We consider mixed economic dependence such that (9) is used for $i \in \{0,1\}$ and (10) is used for $i \in \{2,3\}$.

Table 3: Costs coefficients for each alternative

Cost coefficients κ_i	i = 0	i = 1	i = 2	i = 3
Constant	1.00	1.00	1.00	1.00
Increasing	0.50	0.75	1.25	1.50

The system's inspection rate is selected as $r = 24$ per year and the continuous discount rate is $\alpha = -\ln(0.90)$, corresponding to an annual discount rate of 10%. The soft failure threshold is $F = 1$. We assume $K = 15$ intrinsic age values equally spaced on interval $[0, F]$. In (7), we replace integration with summation and the exponential density function with a truncated geometric probability mass function (see Appendix I). We perform a full factorial experiment leading to $3^4 2^2 = 324$ problem instances.

5.2. Results

We assess the value of the optimal policy by comparing its performance with benchmark usage policies, BUP and UPP. Under each benchmark usage policy, maintenance decisions are optimized. Table 4 reports the average and maximum cost increases associated with BUP and UUP. The cost increase is calculated by $CS = (C - C^*)/C^*$, where C is the expected total discounted cost under the benchmark policy and C^* is the optimal cost obtained by our integrated usage and maintenance optimization model.

Our numerical experiments show that the optimal policy can have a significant value compared to BUP and UUP. The cost increase is up to 8% and 27% under BUP and UUP, respectively. BUP outperforms UUP for most of the problem instances. Although the optimality of BUP cannot be guaranteed when there is positive economic dependence, BUP is usually near-optimal for these cases. On the other hand, UUP performs the worse in case of positive economic dependence.

Counterintuitively, the average cost performance of BUP and UUP are similar in case of negative economic dependence. In addition, none of them is near-optimal. Under BUP, all units would reach the soft failure threshold more or less at the same time. Therefore, the reliability of the whole system decreases over time and simultaneous maintenance of several units gets inevitable. The latter is costly in case of negative economic dependence and its impact increases with the number of units. Under UUP, the intrinsic age of standby units are kept low, making the best use of redundancy. However, always using the oldest units would lead to a situation where maintenance is inevitable in an operating mode in which maintenance costs are high. In order to avoid this, it would be better to activate a unit with low intrinsic age. As such, maintenance of the oldest unit can be postponed to an operating mode in which maintenance costs are cheap. That is why, the cost performance of UUP deteriorates when maintenance costs are increasing and improves when maintenance costs are constant over $i \in I$. For the same reason, the cost performance of BUP deteriorates when maintenance costs are constant and improves when maintenance costs are increasing in $i \in I$. The possibility of postponing maintenance to operating modes in which maintenance costs are cheap can have a significant value. It is not always beneficial to sacrifice this flexibility at the benefit of individual versus simultaneous maintenance.

Table 4: Average and maximum cost increases under BUP and UUP compared to the optimal policy

Parameter	Alternative	BUP		UUP	
		Average	Maximum	Average	Maximum
Number of units	2 units	2.35%	6.44%	6.72%	22.22%
	3 units	3.06%	8.17%	7.60%	26.56%
Operating mode transition rates	Constant	2.72%	7.55%	7.28%	26.56%
	Increasing	2.89%	8.17%	7.87%	24.33%
	Decreasing	2.50%	6.67%	6.31%	26.28%
Degradation rate of an active unit	Constant	2.99%	8.17%	6.71%	21.37%
	Increasing	2.82%	7.14%	3.82%	14.31%
	Decreasing	2.31%	7.13%	10.94%	26.56%
Degradation rate of a standby unit	None	2.70%	8.17%	10.75%	26.56%
	Medium	2.99%	7.97%	7.60%	21.31%
	High	2.43%	6.88%	3.12%	12.59%
Maintenance costs	Constant	3.04%	7.55%	6.58%	26.56%
	Increasing	2.37%	8.17%	7.73%	25.51%
Economic dependency	Positive	0.13%	2.65%	12.17%	26.56%
	Negative	4.85%	8.17%	4.36%	17.14%
	Mixed	3.13%	6.12%	4.94%	19.14%
Overall Average		2.70%		7.16%	

We observe that the cost performances of BUP and UUP are sensitive to the transition and degradation rates. When it is more likely to be in an operating mode in which n units are required (i.e., operating mode transition rates are decreasing), the error that is made because of non-optimal usage decisions is less significant. Contrarily, when it is more likely to be in an operating mode in which $0 < k < n$ (i.e., operating mode transition rates are increasing), the performance of both BUP and UUP decreases. UUP performs worse when the intrinsic age difference of active and stand-by units increases dramatically as a consequence of usage. The latter happens when standby units do not deteriorate and/or the deterioration of the active units is much faster in operating modes where $0 < k < n$ (i.e., degradation rates are decreasing). In contrast, if the deterioration rate of standby units is high, the intrinsic age difference between the oldest and the youngest units would be similar under any usage policy. In this case, both UUP and BUP perform relatively well.

Although BUP is near-optimal for many instances within our full factorial experiment, it is not always a suitable policy. Its performance gets worse for 3-unit systems, under mixed/negative economic dependence, constant maintenance costs, and medium deterioration of standby units. For those cases, optimal usage policy is more valuable.

5.3. An illustrative example

In this section, we illustrate the optimal usage and maintenance policy on an example. This example is inspired by the navy frigates of the RNLN (see Tinga and Janssen, 2013). For the navy frigates, dock maintenance in the home harbor is the cheapest option and maintenance costs are increasing in operating modes due to the costs associated with the transportation of personnel, spare parts, and the resulting downtime. Due to safety concerns and the risk of human error, maintaining more than one unit simultaneously is inadvisable in a foreign harbor (i.e., for i = 1) and during transit and mission states (i.e., for i = 2, 3). On the other hand, there is positive economic dependence in the home harbor (i.e., for i = 0). The transit/mission state that requires 2 active units is the shortest operating mode while docking is the longest operating mode. Active units are considered to degrade with the same rate in different operating modes. The degradation rate of standby units is medium. We consider parameter values presented in Section 5.1, except that the mixed economic dependence is customized. The parameter alternatives considered are detailed in Table 5. Figures 1(a), (b), (c), and (d) present the optimal policy in operating modes i = 0, i = 1, i = 2, and i = 3, respectively.

Table 5: Parameters used in the illustrative example

Parameter	**Alternative**
Number of units	2
Operating mode transition rates	Increasing
Degradation rate of an active unit	Constant
Degradation rate of a standby unit	Medium
Maintenance costs	Increasing
Economic dependence	Customized-Mixed: (9) for i = 0 and (10) for i = {1,2,3}
Cost increase under BUP	6.15%
Cost increase under UUP	6.90%

In Figure 1, the grey areas correspond to the condition values that are excluded from the state space (the solution is symmetrical). Unit 1 represents the old unit and Unit 2 represents the young unit in intrinsic age. For this example, the optimal usage policy is not well-structured, in particular, in operating mode i = 1 and i = 2 (see Figure 1(a) and (b)). The decision of which unit to activate depends on the intrinsic age of both units. Therefore, neither BUP nor UUP is near-optimal. Associated cost increases are 6.15% and 6.90% under BUP and UUP, respectively. There is the following idea behind the optimality of such a usage policy. Depending on the current system condition and operating mode, if maintaining Unit 1 is likely to be optimal as a future action, Unit 1 is activated. On the other hand, if joint maintenance in the home harbor is likely to occur as a future action, Unit 2 is activated. As illustrated by this case, the optimal usage policy is a consequence of the optimal maintenance policy and vice versa.

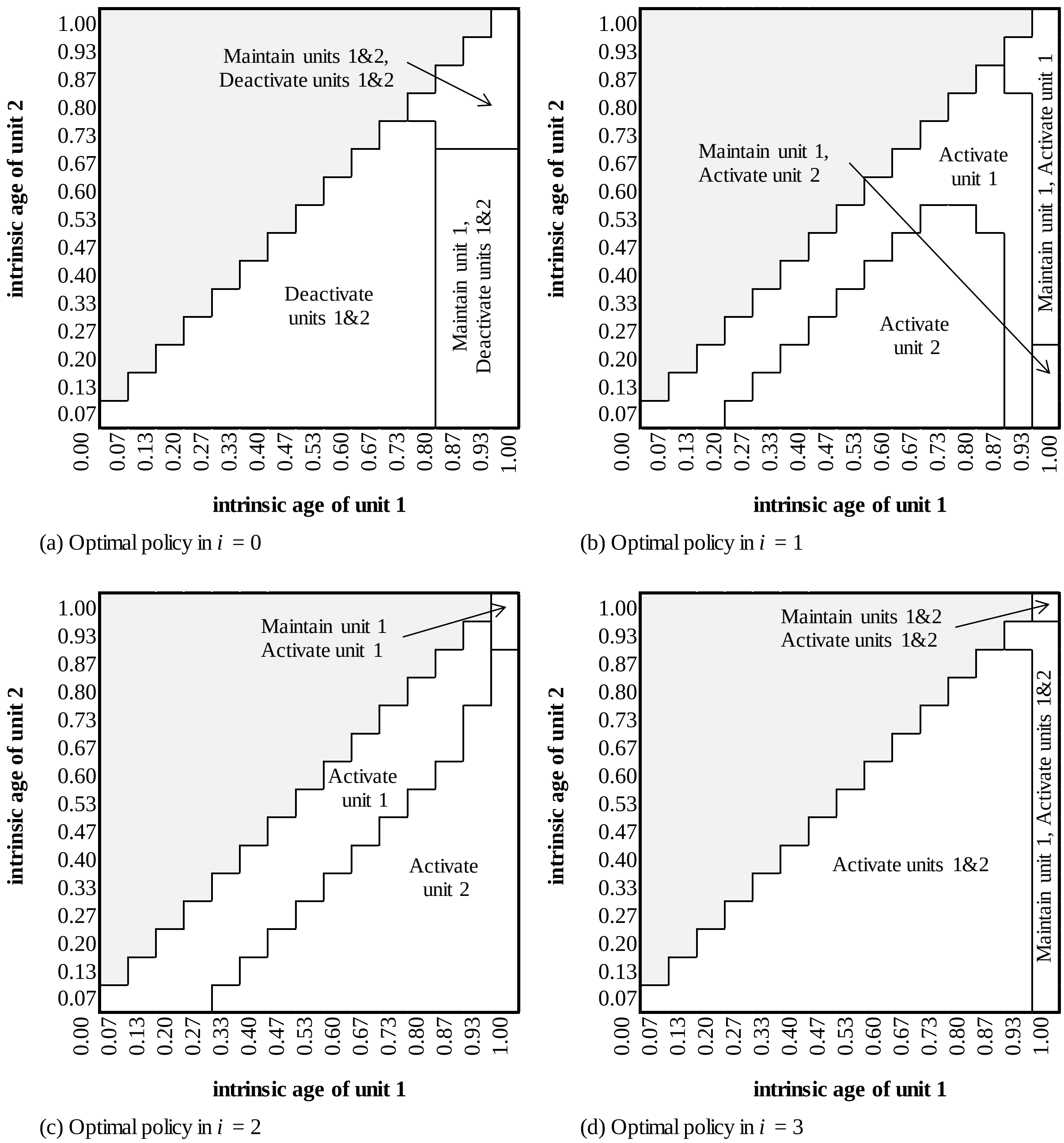

Figure 1: Optimal usage and maintenance policy in operating modes (a) i = 0, (b) i = 1, (c) i = 2, and (d) i = 3

6. Conclusion

In this paper, we considered an integrated usage and maintenance optimization problem for a k-out-of-n system subject to different operating modes which dictate the number of active units, degradation rates of active and standby units, maintenance costs, and the type of economic dependence. We formulated this problem as a Markov decision process. We investigated the structure of the optimal usage and maintenance policy and showed that regardless of the usage policy, it is optimal to maintain the most deteriorated units first. Moreover, the optimality of an all-or-nothing maintenance policy leads to the optimality of a balanced usage policy. Since the optimal usage policy is not obvious in many real-life settings, practitioners operate such *k*-out-of-*n* systems with the balanced usage policy when there is positive economic dependence and with the unbalanced usage policy when there is negative economic dependence. However, the type of economic dependence does not guarantee the optimality of the above-mentioned usage policies. Numerical experiments based on realistic settings showed that, even when the maintenance decisions are optimized, the above-mentioned usage policies are up to 27% more costly than the integrated usage and maintenance policy. Even though there are cases in which these usage policies perform relatively well (e.g., when there is positive economic dependence in all operating modes, the value of *n* is low, and the degradation rate of a standby units is high), there is a significant value of integrating usage and maintenance decisions in many realistic settings, in particular, when the type of economic dependence is not the same in different operating modes (e.g., positive in the home base and negative outside the home base).

The problem studied in this paper was observed for real-life maritime systems. But, our model and results presented can also be applied to other moving assets such as aircrafts and aerospace systems. As presented by Chew et al. (2008), for such systems there exist maintenance-free operation periods during which the systems must be able to carry out all its assigned missions without requiring any maintenance. Following each maintenance-free operation period, there is a period where the moving asset can be maintained. This characteristic can be easily incorporated into our model by introducing maintenance-free operating modes.

In our paper, we have used the intrinsic age concept to model the degradation process of the units within a *k*-out-of-*n* system, some examples being multi-engine propulsion systems, energy generation systems, or refrigeration systems of maritime assets. In order to implement our model, one needs to have the degradation rate of active and standby units in different operating modes. Failure-related historical data is very limited for maritime assets (Eruguz et al., 2015). To model the degradation behavior of such systems, physical degradation models are suitable since they are less data demanding and capable of incorporating the relation between degradation, usage, and environment (see Tinga, 2010). As an outcome of these models, the degradation rate of critical components can be estimated. The intrinsic age of a unit can be assessed by translating the component level degradation information to a unit level condition indicator.

Acknowledgement

This research is funded by Dutch Institute for Advanced Logistics (Dinalog, 2011-4-085-R). The authors gratefully acknowledge the support of Fugro Marine Services and the Royal Netherlands Navy.

Appendix I. Value iteration algorithm

For continuous and infinite state space (i,Ω) the value iteration algorithm is intractable when solving problem (7). For the sake of tractability, the continuous component of the state space can be discretized by $d_s \in \Lambda$ where $\Lambda = \{0, \varepsilon, 2\varepsilon, ..., F\}$, and $\varepsilon \in \Re^+$. We denote the discrete and finite counterparts of system condition vectors Ω and Ω_l by Δ and Δ_l, respectively. In other words, Δ is the set of condition vectors where system units are in descending order of intrinsic ages and Δ_l is a subset of Δ consisting of condition vectors for which exactly $l \in \{0,1,...,n\}$ units are functional.

Since maintenance costs do not depend on system condition vector, in the usage operator $\mathrm{U}_{\mathbf{u},i,x}$ condition levels can be truncated by *F*. We can define the finite counterpart of $\mathrm{U}_{\mathbf{u},i,x}$ as follows:

$$\tilde{\mathrm{U}}_{\mathbf{u},i,x}\,\mathbf{d} = \Theta\left(\mathbf{d} + \sum_{s \in S} \min\{F - d_s, \lambda_i^{u_s} x\}\mathbf{e}_s\right).$$

We define $Z(i,\mathbf{d},\mathbf{u}) = \{0, T_1, T_2, ..., T_{|Z(i,\mathbf{d},\mathbf{u})|}\}$ as the finite set of time intervals after which there is an increase in the intrinsic age of unit(s) in operating mode $i \in I$, for condition vector $\mathbf{d} \in \Delta$, and under usage action $\mathbf{u} \in U(i,\mathbf{d})$. Time interval $T_\tau \in Z(i,\mathbf{d},\mathbf{u})$ satisfies the followings:

(a) $\tilde{\mathrm{U}}_{\mathbf{u},i,T_\tau}\,\mathbf{d} \in \Delta$ for all $T_\tau \in Z(i,\mathbf{d},\mathbf{u})$

(b) $T_\tau < T_{\tau+1}$ for $0 \le \tau < |Z(i,\mathbf{d},\mathbf{u})|$

(c) $T_{|Z(i,\mathbf{d},\mathbf{u})|}$ is the smallest real number that satisfies $\tilde{\mathrm{U}}_{\mathbf{u},i,T_{|Z(i,\mathbf{d},\mathbf{u})|}}\,\mathbf{d} \in \Delta_0$ if $\exists s \in S, \lambda_i^{u_s} > 0$

(d) $T_{|Z(i,\mathbf{d},\mathbf{u})|} = 0$ if $\forall s \in S, \lambda_i^{u_s} = 0$

The corresponding transition probabilities and the discounting factor under continuous discount rate α can be computed as follows:

$$\varphi^{\alpha}_{i,\mathbf{d},\mathbf{u},T_\tau} = \begin{cases} \dfrac{r}{r+\alpha}\left(e^{-(r+\alpha)T_\tau} - e^{-(r+\alpha)T_{\tau+1}}\right) & \text{if } \tau < |Z(i,\mathbf{d},\mathbf{u})| \text{ and } \exists s \in S, \lambda_i^{u_s} > 0 \\ \dfrac{r}{r+\alpha} e^{-(r+\alpha)T_\tau} & \text{if } \tau = |Z(i,\mathbf{d},\mathbf{u})| \text{ and } \exists s \in S, \lambda_i^{u_s} > 0 \\ 1 & \text{if } \forall s \in S, \lambda_i^{u_s} = 0 \end{cases}$$

For $\mathbf{d} \in \Delta$, the discrete counterpart of operator $\Gamma_{\mathbf{m},\mathbf{u}}$ is denoted by $\tilde{\Gamma}_{\mathbf{m},\mathbf{u}}$ and is defined as follows:

$$\tilde{\Gamma}_{\mathbf{m},\mathbf{u}} V(i,\mathbf{d}) = \sum_{j \in I} p_{ij} \sum_{T_\tau \in Z(i,\mathbf{d},\mathbf{u})} \varphi^{\alpha}_{i,\mathbf{d},\mathbf{u},T_\tau} V\left(j, \tilde{\mathrm{U}}_{\mathbf{u},i,T_\tau} \mathrm{M}_{\mathbf{m}}\,\mathbf{d}\right)$$

The discrete and finite state space versions of (7) and (8) can be written as follows for $(i,\mathbf{d}) \in (I,\Delta)$:

$$V_{t+1}(i,\mathbf{d}) = \min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\mathbf{d}),\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\,\mathbf{d})} \left\{ C^{\mathrm{tot}}(i,\sum_{s\in S} m_s) + \tilde{\Gamma}_{\mathbf{m},\mathbf{u}} V_t(i,\mathbf{d}) \right\} \tag{A.1}$$

$$V_0(i,\mathbf{d}) = C^{\mathrm{tot}}\big(i, \max\{k-l, 0\}\big) \text{ for } i \in I_k, \mathbf{d} \in \Delta_l, \text{ and } k,l \in \{0,1,..,n\}. \tag{A.2}$$

Problem (A.1) can be solved numerically using the value iteration algorithm (see Puterman, 2005).

Appendix II. Proof of Theorem 1

Definition A.1: For $\bar{\mathbf{d}}, \underline{\mathbf{d}} \in \Omega$, $\bar{\mathbf{d}} \geq \underline{\mathbf{d}}$ means that $\bar{d}_s \geq \underline{d}_s$ for each $s \in S$.

Lemma A.1: For each $i \in I$ and $t \in \mathbb{N}_0$, $V_t(i,\mathbf{d})$ is increasing in $\mathbf{d} \in \Omega$, i.e.:

$V_t(i,\bar{\mathbf{d}}) - V_t(i,\underline{\mathbf{d}}) \geq 0$ where $\bar{\mathbf{d}}, \underline{\mathbf{d}} \in \Omega$ and $\bar{\mathbf{d}} \geq \underline{\mathbf{d}}$.

Proof of Lemma A.1: We prove Lemma A.1 by induction on $t \in \mathbb{N}_0$.

Basis: For $\bar{\mathbf{d}} \in \Omega_{l_1}$ and $\underline{\mathbf{d}} \in \Omega_{l_2}$ with $l_1, l_2 \in \{0,1,..,n\}$, if $\bar{\mathbf{d}} \geq \underline{\mathbf{d}}$, then $n \geq l_2 \geq l_1 \geq 0$.

From (8), we have $V_0(i,\bar{\mathbf{d}}) - V_0(i,\underline{\mathbf{d}}) \geq 0$. That is, Lemma A.1 holds for $t = 0$.

Induction Step: For each $i \in I$, assume that $V_t(i,\mathbf{d})$ is increasing in $\mathbf{d} \in \Omega$ for a given $t \geq 0$. That is, for $\bar{\mathbf{d}}, \underline{\mathbf{d}} \in \Omega$, if $\bar{\mathbf{d}} \geq \underline{\mathbf{d}}$ then $V_t(i,\bar{\mathbf{d}}) - V_t(i,\underline{\mathbf{d}}) \geq 0$.

For each $i \in I$, the sets of possible maintenance actions satisfy $M(i,\bar{\mathbf{d}}) \subset M(i,\underline{\mathbf{d}})$ (cf. (1)). Similarly, the sets of possible usage actions satisfy $U(i,\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}}) \subset U(i,\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}})$ (cf. (3)).

For action vector $(\mathbf{m},\mathbf{u})$ satisfying $\mathbf{m} \in M(i,\bar{\mathbf{d}})$ and $\mathbf{u} \in U(i,\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}})$, ordering, usage, and maintenance operators (cf. (2) and (4)) preserve monotonicity, i.e.:

$\mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}} \geq \mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}}$ for any time period $x \geq 0$.

From the induction hypothesis:

$V_t\big(i,\mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}}\big) \geq V_t\big(i,\mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}}\big)$ for any time period $x \geq 0$ and for each $i \in I$.

Hence, for each $i \in I$, $\mathbf{m} \in M(i,\bar{\mathbf{d}})$ and $\mathbf{u} \in U(i,\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}})$, we have:

$$\begin{aligned} & C^{\mathrm{tot}}(i,\sum_{s\in S} m_s) + \sum_{j\in I} p_{ij} \int_0^\infty r e^{-(r+\alpha)x} V_t\big(j,\mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}}\big)dx \\ & \geq C^{\mathrm{tot}}(i,\sum_{s\in S} m_s) + \sum_{j\in I} p_{ij} \int_0^\infty r e^{-(r+\alpha)x} V_t\big(j,\mathrm{U}_{\mathbf{u},i,x}\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}}\big)dx \end{aligned},$$

This implies:

$$C^{\text{tot}}(i,\sum_{s\in S} m_s)+\Gamma_{\mathbf{m},\mathbf{u}}V_t(i,\bar{\mathbf{d}})\geq C^{\text{tot}}(i,\sum_{s\in S} m_s)+\Gamma_{\mathbf{m},\mathbf{u}}V_t(i,\underline{\mathbf{d}})$$

Since $M(i,\bar{\mathbf{d}})\subset M(i,\underline{\mathbf{d}})$ and $U(i,\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}})\subset U(i,\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}})$, we have:

$$\begin{aligned}&\min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\bar{\mathbf{d}}),\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\,\bar{\mathbf{d}})}\left\{C^{\text{tot}}(i,\sum_{s\in S} m_s)+\Gamma_{\mathbf{m},\mathbf{u}}V_t(i,\bar{\mathbf{d}})\right\}\\ &\geq \min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\underline{\mathbf{d}}),\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\,\underline{\mathbf{d}})}\left\{C^{\text{tot}}(i,\sum_{s\in S} m_s)+\Gamma_{\mathbf{m},\mathbf{u}}V_t(i,\underline{\mathbf{d}})\right\}\text{for each } i\in I.\end{aligned}\tag{A.3}$$

Thus, $V_{t+1}(i,\bar{\mathbf{d}})\geq V_{t+1}(i,\underline{\mathbf{d}})$ for each $i\in I$, completing the induction on $t\in\mathbb{N}_0$. □

Lemma A.2: For each $i\in I$, $V(i,\mathbf{d})$ is increasing in $\mathbf{d}\in\Omega$.

Proof of Lemma A.2: Since Lemma A.1 holds for all $t\in\mathbb{N}_0$ and the value iteration algorithm convergences to the optimal value (see Corollary 9.17.1 in Bertsekas and Shreve, 2004, p. 235), Lemma A.1 also holds for the infinite horizon function $V(i,\mathbf{d})$. □

Proof of Theorem 1(a): Assume that an optimal maintenance action $\mathbf{m}^*$ in operating mode $i\in I$ and for condition vector $\mathbf{d}\in\Omega$, is such that $\mathbf{m}^*\neq\mathbf{m}_s, \forall s\in\{0,1,\ldots,n\}$. That is, maintenance action vector $\mathbf{m}^*=(m_s^*\,|\,s\in S)$ contains at least two components $m_{s_1}^*=0$ and $m_{s_2}^*=1$, where $s_1,s_2\in S$ and $s_1<s_2$.

Component switching: We obtain maintenance action vector $\tilde{\mathbf{m}}$ by switching the values of the two components $s_1,s_2\in S$ in $\mathbf{m}^*$, i.e., $\tilde{m}_{s_1}=1$, $\tilde{m}_{s_2}=0$, and $\tilde{m}_s=m_s^*$ for $s\in S$, $s\neq s_1$, and $s\neq s_2$.

Maintenance actions $\mathbf{m}^*$ and $\tilde{\mathbf{m}}$ satisfy:

$$\sum_{s\in S} m^*=\sum_{s\in S}\tilde{m}.$$

From (1), $\mathbf{m}^*\in M(i,\mathbf{d})$ implies $\tilde{\mathbf{m}}\in M(i,\mathbf{d})$. By the definition of $\mathbf{d}\in\Omega$, $d_{s_1}\geq d_{s_2}$. Thus, we have:

$$\mathrm{M}_{\mathbf{m}^*}\,\mathbf{d}\geq\mathrm{M}_{\tilde{\mathbf{m}}}\,\mathbf{d},$$

implying $U(i,\mathrm{M}_{\mathbf{m}^*}\,\mathbf{d})\subset U(i,\mathrm{M}_{\tilde{\mathbf{m}}}\,\mathbf{d})$. In addition, from Lemma A.2, we have:

$$V(i,\mathrm{M}_{\mathbf{m}^*}\,\mathbf{d})\geq V(i,\mathrm{M}_{\tilde{\mathbf{m}}}\,\mathbf{d}).$$

Thus:

$$\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}^*}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,\sum_{s\in S}m_s^*)+\Gamma_{\mathbf{m}_0,\mathbf{u}}V(i,\mathrm{M}_{\mathbf{m}^*}\mathbf{d})\right\}$$
$$\geq \min_{\mathbf{u}\in U(i,\mathrm{M}_{\tilde{\mathbf{m}}}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,\sum_{s\in S}\tilde{m}_s)+\Gamma_{\mathbf{m}_0,\mathbf{u}}V(i,\mathrm{M}_{\tilde{\mathbf{m}}}\mathbf{d})\right\} \quad \text{for each } i\in I.$$

By the definition of $\Gamma_{\mathbf{m},\mathbf{u}}$ (cf. (6)):

$\Gamma_{\mathbf{m}_0,\mathbf{u}}V(i,\mathrm{M}_{\mathbf{m}}\mathbf{d})=\Gamma_{\mathbf{m},\mathbf{u}}V(i,\mathbf{d})$ for each $\mathbf{m}\in M(i,\mathbf{d})$ and $\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\mathbf{d})$.

Therefore, for $\mathbf{u}^*\in U(i,\mathrm{M}_{\mathbf{m}^*}\mathbf{d})$ satisfying:

$$C^{\mathrm{tot}}(i,\sum_{s\in S}m_s^*)+\Gamma_{\mathbf{m}^*,\mathbf{u}^*}V(i,\mathbf{d})=\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}^*}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,\sum_{s\in S}m_s^*)+\Gamma_{\mathbf{m}^*,\mathbf{u}}V(i,\mathbf{d})\right\}$$

There exist $\tilde{\mathbf{u}}\in U(i,\mathrm{M}_{\tilde{\mathbf{m}}}\mathbf{d})$ such that:

$$C^{\mathrm{tot}}(i,\sum_{s\in S}m_s^*)+\Gamma_{\mathbf{m}^*,\mathbf{u}^*}V(i,\mathbf{d})\geq C^{\mathrm{tot}}(i,\sum_{s\in S}\tilde{m}_s)+\Gamma_{\tilde{\mathbf{m}},\tilde{\mathbf{u}}}V(i,\mathbf{d})$$

Thus, in state $(i,\mathbf{d})\in(I,\Omega)$, action $(\tilde{\mathbf{m}},\tilde{\mathbf{u}})$ is at least equally good action as $(\mathbf{m}^*,\mathbf{u}^*)$ in terms of the expected total discounted maintenance cost. Using the same logic, i.e., by applying *component switching* multiple times, we can obtain an optimal action vector $(\mathbf{m}_s,\mathbf{u})$. If $(i,\mathbf{d})\in I_k\times\Omega_l$ with $k,l\in\{0,1,\ldots,n\}$, action $\mathbf{m}_s$ is a possible actions, i.e., $\mathbf{m}_s\in M(i,\mathbf{d})$, if and only if $s\in\{\max\{0,k-l\},\ldots,n\}$ (cf. (1)). □

Proof of Theorem 1(b): Assume that maintenance policy $\mathbf{m}_s$ is optimal in $(i,\mathbf{d})\in(I,\Omega)$, i.e.:

$$V(i,\mathbf{d})=\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,s)+\Gamma_{\mathbf{m}_s,\mathbf{u}}V(i,\mathbf{d})\right\}$$

From Lemma A.2, for $\varepsilon_{s'}\in\Re^+$ and $s'\in S$, we have:

$$V(i,\mathbf{d})\leq V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'})) \tag{A.4}$$

Moreover, by the definition of operator $\mathrm{M}_{\mathbf{m}_s}$, we have:

$$\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,s)+\Gamma_{\mathbf{m}_s,\mathbf{u}}V(i,\mathbf{d})\right\}=\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,s)+\Gamma_{\mathbf{m}_s,\mathbf{u}}V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'})\right\} \text{ for } s'\leq s.$$

By the definition of the value function:

$$V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'}))\leq \min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,s)+\Gamma_{\mathbf{m}_s,\mathbf{u}}V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'})\right\} \text{ for } s'\in S$$

Implying that:

$V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'}))\leq V(i,\mathbf{d})$ for $s'\leq s$. (A.5)

From (A.4) and (A.5):

$$V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'}))=V(i,\mathbf{d})=\min_{\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d})}\left\{C^{\mathrm{tot}}(i,s)+\Gamma_{\mathbf{m}_s,\mathbf{u}}V(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'})\right\}\text{ for } s'\leq s.$$

Therefore, $\mathbf{m}_s$ is also optimal for $(i,\Theta(\mathbf{d}+\varepsilon_{s'}\mathbf{e}_{s'}))\in I\times\Omega$ and $s'\leq s$. □

Appendix III. Proof of Theorem 2

Lemma A.3: If either $\mathbf{m}_0$ or $\mathbf{m}_n$ is optimal in each state $(i,\mathbf{d})\in I\times\Omega$ for all *t*-stage problems where $t\in\mathbb{N}_0\backslash\{0\}$, then $V_t(i,\mathbf{d})$ is upstream increasing in $\mathbf{d}\in\Omega$ for all $t\in\mathbb{N}_0$, i.e.:

$$V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s))-V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1}))\geq 0 \text{ for all } s\in S\setminus\{n\},\ \varepsilon\in\Re^+, \text{ and } d_s<F.$$

Proof of Lemma A.3: First, we can show that:

For $\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s)\in\Omega_{l_1}$ and $\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})\in\Omega_{l_2}$ with $l_1,l_2\in\{0,1,..n\}$, we have $l_1\leq l_2$. (A.6)

This is because:

- If $d_s+\varepsilon<F$, then $d_{s+1}+\varepsilon<F$ and $l_1=l_2$.
- If $d_s+\varepsilon\geq F$ and $d_{s+1}+\varepsilon<F$, then $l_1\leq l_2$.
- If $d_s+\varepsilon\geq F$ and $d_{s+1}+\varepsilon\geq F$, then $l_1=l_2$ (since $d_{s+1}\leq d_s<F$).

Next, we prove Lemma A.3 by induction on $t\in\mathbb{N}_{0.}$

Basis: Consider (8) for $t=0$. For $i\in I_k$ where $k\in\{0,1,..n\}$, from (A.6):

$$V_0(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s))=C^{\mathrm{tot}}(i,\max\{k-l_1,0\})\geq C^{\mathrm{tot}}(i,\max\{k-l_2,0\})=V_0(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})).$$

Therefore, Lemma A.3 holds for $t=0$.

Induction Step: For each $i\in I$, assume that $V_t(i,\mathbf{d})$ is upstream increasing in $\mathbf{d}\in\Omega$ for a given $t\geq 0$, , i.e.:

$$V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s))-V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1}))\geq 0 \text{ for all } s\in S\setminus\{n\},\ \varepsilon\in\Re^+, \text{ and } d_s<F.$$

As a given condition in Lemma A.3, the set of optimal maintenance decisions $M^*(i,\mathbf{d})\subset\{\mathbf{m}_0,\mathbf{m}_n\}$ in each state $(i,\mathbf{d})\in I\times\Omega$ and for all *t*-stage problems where $t\in\mathbb{N}_0\backslash\{0\}$.

First, consider action $\mathbf{m}_n$. By definition, we have:

$$\mathrm{M}_{\mathbf{m}_n}\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s)=\mathrm{M}_{\mathbf{m}_n}\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})=\mathbf{e}_0$$

where $\mathbf{e}_0$ is the null vector. Therefore, for each usage action $\mathbf{u} \in U(i, \mathbf{e}_0)$ we have:

$$\Gamma_{\mathbf{m}_n,\mathbf{u}} V_t(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)) = \Gamma_{\mathbf{m}_n,\mathbf{u}} V_t(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1})) \tag{A.7}$$

Second, consider action $\mathbf{m}_0$. We have the following regarding maintenance operator $\mathrm{M}_{\mathbf{m}_0}$:

$\mathrm{M}_{\mathbf{m}_0} \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s) = \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)$ and $\mathrm{M}_{\mathbf{m}_0} \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1}) = \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1})$

If $\mathbf{m}_0 \in M(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s))$, then $\mathbf{m}_0 \in M(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1}))$ since $d_{s+1} \le d_s < F$. From (A.6), we have:

$$U(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)) \subset U(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1})). \tag{A.8}$$

In addition, from (A.6):

$$M(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)) \subset M(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1})). \tag{A.9}$$

From the induction hypothesis and (A.7), (A.8) and (A.9):

$$\min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s)),\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s))} \left\{ C^{\mathrm{tot}}(i, \sum_{s\in S} m_s) + \Gamma_{\mathbf{m},\mathbf{u}} V_t(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)) \right\}$$
$$\ge \min_{(\mathbf{m},\mathbf{u})|\mathbf{m}\in M(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})),\mathbf{u}\in U(i,\mathrm{M}_{\mathbf{m}}\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1}))} \left\{ C^{\mathrm{tot}}(i, \sum_{s\in S} m_s) + \Gamma_{\mathbf{m},\mathbf{u}} V_t(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1})) \right\} \text{ for each } i \in I.$$

Thus:

$V_{t+1}(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_s)) \ge V_{t+1}(i, \Theta(\mathbf{d} + \varepsilon \mathbf{e}_{s+1}))$ for each $i \in I$,

completing the induction on $t \in \mathbb{N}_0$. □

Lemma A.4: If either $\mathbf{m}_0$ or $\mathbf{m}_n$ is optimal in each state $(i, \mathbf{d}) \in I \times \Omega$ for all *t*-stage problems where $t \in \mathbb{N}_0 \backslash \{0\}$, then $V(i, \mathbf{d})$ is upstream increasing in $\mathbf{d} \in \Omega$.

Proof of Lemma A.4: Since Lemma A.3 holds for all $t \in \mathbb{N}_0$ and the value iteration algorithm convergences to the optimal value (see Corollary 9.17.1 in Bertsekas and Shreve, 2004, p. 235), Lemma A.3 also holds for the infinite horizon function $V(i, \mathbf{d})$. □

Proof of Theorem 2: From Lemma A.3, an increase in a downstream component of condition vector $\mathbf{d} \in \Omega$ is preferable (in terms of lower expected cost) to an increase in an upstream yet functional component of $\mathbf{d} \in \Omega$ in all *t*-stage problems where $t \in \mathbb{N}_0$. From Lemma A.4, the latter also holds for the infinite horizon problem. Therefore, since $\lambda_i^1 \ge \lambda_i^0$, it is preferable to activate *k* lowest deteriorated units in operating mode $i \in I_k$. In other words, the associated optimal usage decision corresponds to $\mathbf{u}_{n-k+1,n}$ in each operating mode $i \in I_k$ where $k \ge 1$ for all *t*-stage problems and for the infinite horizon problem. □

Appendix IV. Remarks on positive and negative economic dependence

Remark 1: For the optimality of an all-or-nothing maintenance policy, positive economic dependence while maintaining *n* units is necessary in all operating modes.

Proof of Remark 1: Under an all-or-nothing maintenance policy, in each state $(i,\mathbf{d}) \in I_k \times \Omega_l$ where $k,l \in \{0,1,\ldots,n\}$ and $k-l>0,$ the cost of maintaining *n* units is cheaper than the cost of maintaining *s* units for all $s \in \{k-l,\ldots,n\}$. (We note that $\mathbf{m}_s$ is not a feasible decision if $s<k-l$). That is, for a certain $t \in \mathbb{N}_0$, if $(i,\mathbf{d}) \in I_k \times \Omega_l$ and $k-l>0,$ we have:

$$V_{t+2}(i,\mathbf{d}) = C^{\text{tot}}(i,n) + \min_{\mathbf{u}|\mathbf{u}\in U\left(i,\mathbf{M}_{\mathbf{m}_n}\mathbf{d}\right)} \left\{\Gamma_{\mathbf{m}_n,\mathbf{u}} V_{t+1}(i,\mathbf{d})\right\}. \quad \text{(A.10)}$$

In addition:

$$C^{\text{tot}}(i,n) + \min_{\mathbf{u}|\mathbf{u}\in U\left(i,\mathbf{M}_{\mathbf{m}_n}\mathbf{d}\right)} \left\{\Gamma_{\mathbf{m}_n,\mathbf{u}} V_{t+1}(i,\mathbf{d})\right\} \le C^{\text{tot}}(i,s) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{M}_{\mathbf{m}_s}\mathbf{d})} \left\{\Gamma_{\mathbf{m}_s,\mathbf{u}} V_{t+1}(i,\mathbf{d})\right\} \quad \text{(A.11)}$$

for all $s \in \{k-l,\ldots,n\}$

(A.11) is equivalent to:

$$C^{\text{tot}}(i,n) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{e}_0)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}(i,\mathbf{e}_0)\right\} \le C^{\text{tot}}(i,s) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{M}_{\mathbf{m}_s}\mathbf{d})} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}\left(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d}\right)\right\} \quad \text{(A.12)}$$

for all $s \in \{k-l,\ldots,n\}$

By the definition of the value function, we have the following:

$$V_{t+1}\left(i,\mathrm{M}_{\mathbf{m}_s}\mathbf{d}\right) \le C^{\text{tot}}(i,n-s) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{e}_0)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_t(i,\mathbf{e}_0)\right\} \text{ for all } s \in \{k-l,\ldots,n\}$$

Since the system's condition is perfect when $\mathbf{d}=\mathbf{e}_0$, we have:

$$V_{t+1}(i,\mathbf{e}_0) = \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{e}_0)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_t(i,\mathbf{e}_0)\right\}.$$

After re-arranging (A.12), we obtain:

$$\begin{aligned} &C^{\text{tot}}(i,n) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{e}_0)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}(i,\mathbf{e}_0)\right\} \\ &\le C^{\text{tot}}(i,s) + C^{\text{tot}}(i,n-s) + \min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{M}_{\mathbf{m}_s}\mathbf{d})} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}(i,\mathbf{e}_0)\right\} \text{ for all } s \in \{k-l,\ldots,n\} \end{aligned} \quad \text{(A.13)}$$

From Theorem 2, we have:

$$\min_{\mathbf{u}|\mathbf{u}\in U(i,\mathbf{e}_0)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}(i,\mathbf{e}_0)\right\} = \min_{\mathbf{u}|\mathbf{u}\in U\left(i,\mathbf{M}_{\mathbf{m}_s}\mathbf{d}\right)} \left\{\Gamma_{\mathbf{m}_0,\mathbf{u}} V_{t+1}(i,\mathbf{e}_0)\right\} = \Gamma_{\mathbf{m}_0,\mathbf{u}_{n-k+1,n}} V_{t+1}(i,\mathbf{e}_0)$$

Thus, inequality (A.13) holds if and only if:

$$C^{\text{tot}}(i,n) \le C^{\text{tot}}(i,n-s) + C^{\text{tot}}(i,s) \text{ for all } s \in \{k-l,...,n\} \tag{A.14}$$

Under an all-or-nothing policy, (A.10) holds for each state $(i,\mathbf{d}) \in I_k \times \Omega_l$ where $k,l \in \{0,1,...,n\}$ and $k-l > 0.$ Hence, the value of *s* varies between 1 and *n* and (A.14) is equivalent to positive economic dependence while maintaining *n* units. □

Remark 2: The optimality of UUP cannot be guaranteed as a counterpart of Theorem 2, i.e., under the optimality of a certain maintenance policy.

Proof Remark 2: In state $(i,\mathbf{d}) \in I \times \Omega$, the counterpart of Lemma A.3 is:

$$V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s)) - V_t(i,\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})) \le 0 \text{ for all } t \in \mathbb{N}_0,\ s \in S \setminus \{n\},\ \varepsilon \in \Re_+, \text{ and } d_s < F. \tag{A.15}$$

Inequality (A.15) implies that $V_t(i,\mathbf{d})$ is upstream decreasing in $\mathbf{d} \in \Omega.$ The latter should hold in each state $(i,\mathbf{d}) \in I \times \Omega$ under a given maintenance policy, so that the counterpart of Theorem 2 holds. Along the same lines as the proof Lemma A.3, (A.15) cannot be guaranteed for a given maintenance policy because of (A.9). (A.9) states that in operating mode $i \in I,$ the number of possible maintenance decisions for condition vector $\Theta(\mathbf{d}+\varepsilon\mathbf{e}_s)$ is less than or equal to the number of possible maintenance decisions for action vector $\Theta(\mathbf{d}+\varepsilon\mathbf{e}_{s+1})$. Indeed, if an additional deterioration of $\varepsilon > 0$ leads to a soft failure of an old unit, the same additional deterioration does not necessarily lead to a soft failure of a young unit. As a consequence, one cannot guarantee that an increase in upstream components of condition vector $\mathbf{d} \in \Omega$ would be preferable (in terms of lower expected cost) to an increase in downstream components of $\mathbf{d} \in \Omega$. Thus, the optimality of UUP cannot be guaranteed as a counterpart of Theorem 2. □

Remark 3: Negative economic dependence in all operating modes is neither necessary nor sufficient condition for the optimality of UUP.

Proof of Remark 3: Table 4 shows that negative economic dependence is not sufficient for the optimality of UUP. The following counter-example shows that negative economic dependence is not a necessary condition either.

We consider parameter values presented in Section 5.1 (see Table A.1 and Figure A.1). The degradation rate of an active unit is set by $\lambda_0^1 = 0,$ $\lambda_i^1 = 3.0$ for $i \in \{1,2\},$ and $\lambda_3^1 = 0.5.$ This represents a situation in which the activation of a single unit causes a much higher deterioration than the activation of both units. Maintenance costs are constant over operating modes and economic dependence is positive in all operating modes.

Table A.1: Parameters used in the counter-example

Parameter	Alternative
Number of units	2
Operating mode transition rates	Decreasing
Degradation rate of an active unit	*Customized*
Degradation rate of a standby unit	High
Maintenance costs	Constant
Economic dependence	Positive
Cost increase under BUP	3.59%
Cost increase under BUP	None (Optimal)

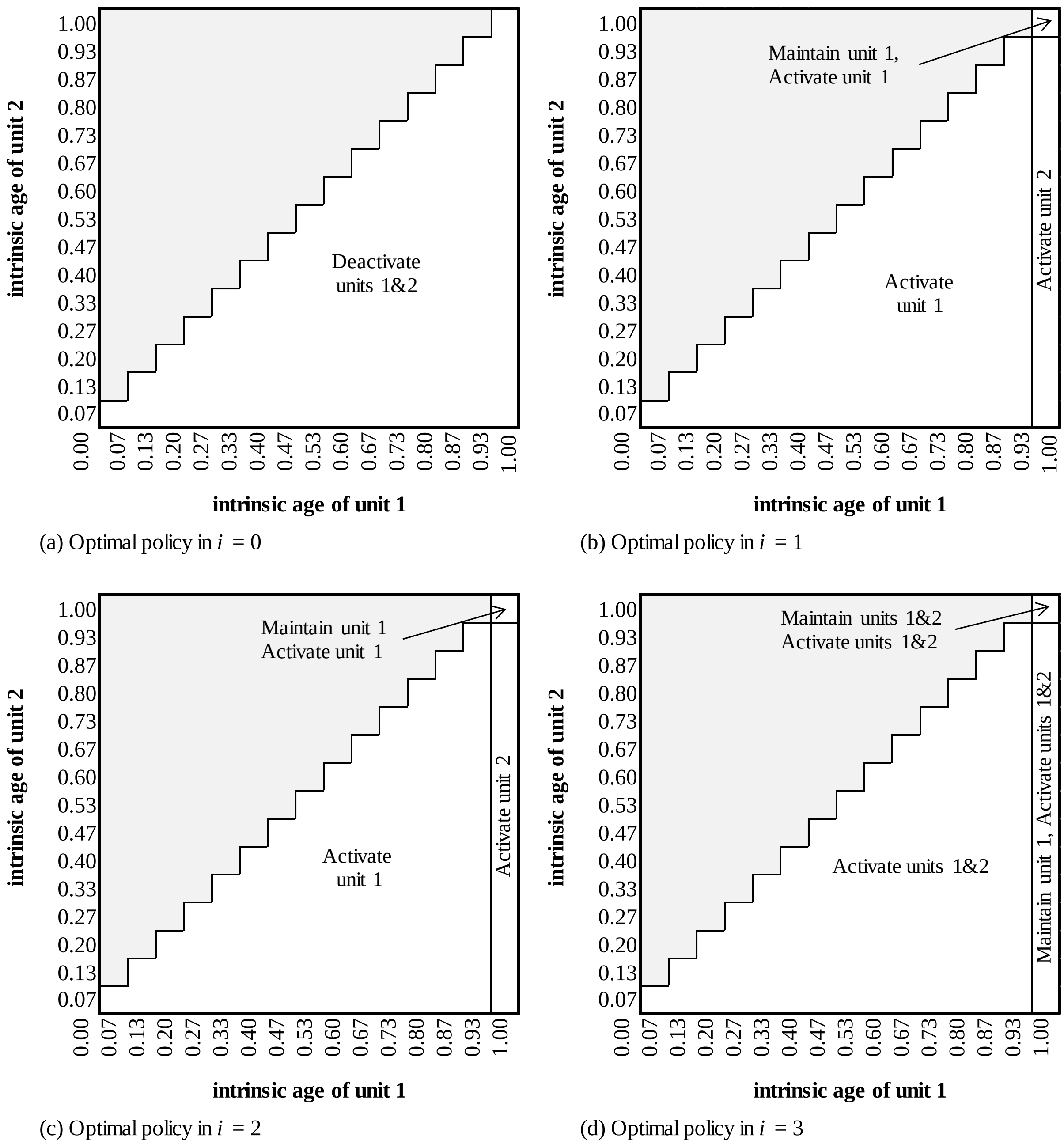

Figure A.1: Optimal usage and maintenance policy in operating modes (a) i = 0, (b) i = 1, (c) i = 2, and (d) i = 3 in the counter-example of Remark 2

As presented in Figures A.1, the optimal policy is a corrective maintenance policy combined with UUP. We observe that in the optimal policy, there is no incentive to maintain the units preventively despite positive economic dependence. After crossing the soft failure threshold, the failed unit is switched-off and its maintenance is postponed until observing $i \in \{3\}$ or $i \in \{1,2\}$ and $d_2 \geq F$. □

References

Alam, M., Al-Saggaf, U.M., 1986. Quantitative reliability evaluation of repairable phased-mission systems using Markov approach. IEEE Trans. Rel. 35, 498–503. doi:10.1109/TR.1986.4335529

Bertsekas, D.P., Shreve, S., 2004. Stochastic Optimal Control: The Discrete-Time Case. New York, NY: Academic Press.

Bouvard, K., Artus, S., Bérenguer, C., Cocquempot, V., 2011. Condition-based dynamic maintenance operations planning & grouping. Application to commercial heavy vehicles. Reliab. Eng. Syst. Safe., ESREL 2009 Special Issue 96, 601–610. doi:10.1016/j.ress.2010.11.009

Bricker, J.L., 1973. A unified method for analyzing mission reliability for fault tolerant computer systems. IEEE Trans. Rel. R-22, 72–77. doi:10.1109/TR.1973.5216037

Çekyay, B., Özekici, S., 2015. Optimal maintenance of semi-Markov missions. Probab. Eng. Inform. Sc. 29, 77–98. doi:10.1017/S0269964814000229

Çekyay, B., Özekici, S., 2012. Optimal maintenance of systems with Markovian mission and deterioration. Eur. J. Oper. Res. 219, 123–133. doi:10.1016/j.ejor.2011.12.037

Chew, S.P., Dunnett, S.J., Andrews, J.D., 2008. Phased mission modelling of systems with maintenance-free operating periods using simulated Petri nets. Reliab. Eng. Syst. Safe., Bayesian Networks in Dependability 93, 980–994. doi:10.1016/j.ress.2007.06.001

Çinlar, E., Özekici, S., 1987. Reliability of Complex Devices in Random Environments. Probab. Eng. Inform. Sc. 1, 97–115. doi:10.1017/S0269964800000322

Dai, Y., Levitin, G., Xing, L., 2016. Optimal periodic inspections and activation sequencing policy in standby systems with condition-based mode transfer. IEEE Trans. Rel. PP, 1–13. doi:10.1109/TR.2015.2495175

Dekker, R., Wildeman, R.E., Schouten, F.A. van der D., 1997. A review of multi-component maintenance models with economic dependence. Math. Method Oper. Res. 45, 411–435. doi:10.1007/BF01194788

Eruguz, A.S., Tan, T., Van Houtum, G.J., 2016. Integrated maintenance and spare part optimization for moving assets, BETA publicatie : working papers, No. 500. Eindhoven University of Technology, https://pure.tue.nl/ws/files/19476959/wp_500.pdf.

Eruguz, A.S., Tan, T., Van Houtum, G.J., 2015. A survey of maintenance and service logistics management : classification and research agenda from a maritime sector perspective., BETA publicatie : working papers, No. 489. Eindhoven University of Technology, https://pure.tue.nl/ws/files/3815115/24194911834760.pdf.

Esary, J.D., Ziehms, H., 1975. Reliability analysis of phased missions, in: Proceedings of the Conference on Reliability and Fault Tree Analysis. Presented at the SIAM, pp. 213–236.

Flory, J.A., Kharoufeh, J.P., Abdul-Malak, D.T., 2015. Optimal replacement of continuously degrading systems in partially observed environments. Nav. Res. Log. 62, 395–415. doi:10.1002/nav.21638

Huynh, K.T., Barros, A., Bérenguer, C., 2015. Multi-level decision-making for the predictive maintenance of k-out-of-n: F Deteriorating Systems. IEEE Trans. Rel. 64, 94–117. doi:10.1109/TR.2014.2337791

Kharoufeh, J.P., 2003. Explicit results for wear processes in a Markovian environment. Oper. Res. Lett. 31, 237–244. doi:10.1016/S0167-6377(02)00229-8

Kharoufeh, J.P., Cox, S.M., 2005. Stochastic models for degradation-based reliability. IIE Trans. 37, 533–542. doi:10.1080/07408170590929009

Kharoufeh, J.P., Solo, C.J., Ulukus, M.Y., 2010. Semi-Markov models for degradation-based reliability. IIE Trans. 42, 599–612. doi:10.1080/07408170903394371

Kim, K., Park, K.S., 1994. Phased-mission system reliability under Markov environment. IEEE Trans. Rel. 43, 301–309. doi:10.1109/24.295013

Levitin, G., Xing, L., Dai, Y., 2014. Minimum mission cost cold-standby sequencing in non-repairable multi-phase systems. IEEE Trans. Rel. 63, 251–258. doi:10.1109/TR.2014.2299192

Levitin, G., Xing, L., Zonouz, A.E., Dai, Y., 2016. Heterogeneous warm standby multi-phase systems with variable mission time. IEEE Trans. Rel. 65, 381–393. doi:10.1109/TR.2015.2451271

Nicolai, R.P., Dekker, R., 2008. Optimal maintenance of multi-component systems: A review, in: Complex System Maintenance Handbook, Springer Series in Reliability Engineering. Springer London, pp. 263–286.

Olde Keizer, M.C.A., Teunter, R.H., Veldman, J., 2016. Clustering condition-based maintenance for systems with redundancy and economic dependencies. Eur. J. Oper. Res. 251, 531–540. doi:10.1016/j.ejor.2015.11.008

Özekici, S., 1995. Optimal maintenance policies in random environments. Eur. J. Oper. Res., OR Models for Maintenance Management and Quality Control 82, 283–294. doi:10.1016/0377-2217(94)00264-D

Özekici, S., Soyer, R., 2004. Reliability modeling and analysis in random environments, in: Soyer, R., Mazzuchi, T.A., Singpurwalla, N.D. (Eds.), Mathematical Reliability: An Expository Perspective, International Series in Operations Research & Management Science. Springer US, pp. 249–273. doi:10.1007/978-1-4419-9021-1_13

Pham, H., Wang, H., 2000. Optimal (τ, T) opportunistic maintenance of a k-out-of-n:G system with imperfect PM and partial failure. Nav. Res. Log. 47, 223–239. doi:10.1002/(SICI)1520-6750(200004)47:3<223::AID-NAV3>3.0.CO;2-A

Puterman, M.L., 2005. Markov Decision Processes: Discrete Stochastic Dynamic Programming, 1 edition. ed. Wiley-Interscience, Hoboken, NJ.

Rokseth, B., Utne, I.B., 2015. Dynamic risk assessment of marine systems, in: Podofillini, L., Sudret, B., Stojadinovic, B., Zio, E., Kröger, W. (Eds.), Proceedings of the 25th European Safety and Reliability Conference, ESREL 2015. CRC Press, Zurich, Switzerland, pp. 725–733.

Sheu, S.-H., Kuo, C.-M., 1994. Optimization problems in k-out-of-n systems with minimal repair. Reliab. Eng. Syst. Safe. 44, 77–82. doi:10.1016/0951-8320(94)90108-2

Tinga, T., 2010. Application of physical failure models to enable usage and load based maintenance. Reliab. Eng. Syst. Safe. 95, 1061–1075. doi:10.1016/j.ress.2010.04.015

Tinga, T., Janssen, R., 2013. The interplay between deployment and optimal maintenance intervals for complex multi-component systems. Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability 227, 227–240. doi:10.1177/1748006X13480743

Tinsley, D., 2016. Condition-based maintenance: A time and money saver. Marine Maintenance Technology International 20–25.

Ulukus, M.Y., Kharoufeh, J.P., Maillart, L.M., 2012. Optimal replacement policies under environment-driven degradation. Probab. Eng. Inform. Sc. 26, 405–424. doi:10.1017/S0269964812000083

Vu, H.C., Do, P., Barros, A., Bérenguer, C., 2014. Maintenance grouping strategy for multi-component systems with dynamic contexts. Reliab. Eng. Syst. Safe. 132, 233–249. doi:10.1016/j.ress.2014.08.002

Wildeman, R.E., Dekker, R., Smit, A.C.J.M., 1997. A dynamic policy for grouping maintenance activities. Eur. J. Oper. Res, Eleventh EURO Summer Institute: Operational Research Models in Maintenance 99, 530–551. doi:10.1016/S0377-2217(97)00319-6

Winokur, H.S., Goldstein, L.J., 1969. Analysis of mission-oriented systems. IEEE Trans. Rel. R-18, 144–148. doi:10.1109/TR.1969.5216341

Xiang, Y., Cassady, C.R., Pohl, E.A., 2012. Optimal maintenance policies for systems subject to a Markovian operating environment. Comput. Ind. Eng. 62, 190–197. doi:10.1016/j.cie.2011.09.006

Xing, L., Amari, S.V., Wang, C., 2012. Reliability of k-out-of-n systems with phased-mission requirements and imperfect fault coverage. Reliab. Eng. Syst. Safe. 103, 45–50. doi:10.1016/j.ress.2012.03.018

Zhang, Z., Wu, S., Li, B., Lee, S., 2013. Optimal maintenance policy for multi-component systems under Markovian environment changes. Expert Syst. Appl. 40, 7391–7399. doi:10.1016/j.eswa.2013.07.003

Zhu, Q., Peng, H., Houtum, G.-J. van, 2015. A condition-based maintenance policy for multi-component systems with a high maintenance setup cost. OR Spectrum 37, 1007–1035. doi:10.1007/s00291-015-0405-z