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Beta Working Paper series 510

BETA publicatie	WP 510 (working paper)
ISBN	
ISSN	
NUR	804
Eindhoven	July 2016

A note on Maximal Covering Location Games

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July 5, 2016

Abstract

In this note we introduce and analyse maximal covering location games. As the core may be empty, several sufficient conditions for core non-emptiness are presented. For each condition we provide an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

1 Introduction

In the maximal covering location problem (Church and ReVelle [1]) a single decision maker has to position a predetermined number of resources in order to maximize profit of the *covered* demand points, where a demand point is covered if a resource is positioned within a certain radius. This well-known location model has proven to be useful in many settings, e.g., for positioning of emergency vehicles (Li et al. [4]), cell towers (Lee and Murray [3]) and retail stores (Plastria and Vanhaverbeke [5]). Another interesting setting is the one with several small-sized regions, e.g., villages or municipalities, that each may or may not own a *single* resource to cover their region completely. When those regions pool their resources a maximal covering location problem arises. Typically, additional coverage, and so additional profit, can be realized and a *joint profit allocation* issue arises amongst the collaborating regions. In this note, we will investigate this allocation aspect. We introduce a maximal covering location (MCL) situation wherein regions are represented by *single* demand points that may or may not keep a *single* resource. For such an MCL situation, an associated MCL game is introduced. For this game, we provide several sufficient conditions (in terms of the

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number of players, the number of resources, and the underlying integer linear program) for core non-emptiness. For each condition we provide an example showing that when the condition is not satisfied, core non-emptiness is not guaranteed.

The outline of this note is as follows. We start in Section 2 with preliminaries on cooperative game theory. In Section 3, we introduce MCL situations, subsequently we introduce MCL games, and finally we present our results.

2 Preliminaries

In this section, we provide some basic elements of cooperative game theory. Consider a finite set $N = \{1, 2, ..., n\}$ of *players* and a function $v : 2^N \to \mathbb{R}$ called the *characteristic function*, with $v(\emptyset) = 0$. The pair (N, v) is a *cooperative game with transferable utility*, shortly *game*. A subset $S \subseteq N$ is a *coalition* and $v(S)$ is the worth that coalition S can obtain by itself. The worth can be transferred freely among the players. The set N is the *grand coalition*. A game (N, v) is *superadditive* if the value of the union of any two disjoint coalitions is larger than or equal to the sum of the values of these disjoint coalitions, i.e., $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$ and *monotonic* if the value of every coalition is at least the value of any of its subcoalitions, i.e., $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. An allocation for a game (N, v) is an n-dimensional vector $x \in \mathbb{R}^N$ where player $i \in N$ receives x_i. An allocation is *efficient* if $\sum_{i \in N} x_i = v(N)$. This implies that all worth is divided among the players of the grand coalition N. An allocation is *stable* if no group of players has an incentive to leave the grand coalition N, i.e., $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$. The set of efficient and stable allocations of (N, v) is the *core* of (N, v) and denoted by $C(N, v)$.

3 Model

In this section, we introduce maximal covering location situations and define the associated games, called maximal covering location games. Finally, we present properties of maximal covering location games.

3.1 Maximal covering location situation

We consider an environment with a finite set $N = \{1, 2, \ldots, n\}$ of players and a finite set $L = \{n + 1, n + 2, \ldots, n + l\}$ of possible resource locations. The distance between player $i \in N$ and resource location $j \in L$ is denoted by $d_{ij} \in \mathbb{R}_+$. For any player $i \in N$, we introduce $r_i \in \{0,1\}$, where $r_i = 1$ indicates that player $i \in N$ owns a resource, and $r_i = 0$ indicates that player $i \in N$ does *not* own a resource. Every player $i \in N$ with $r_i = 1$ positions its resource at any resource location $j \in L$ and if $d_{ij} \leq \mathscr{D} \in \mathbb{R}_+$, i.e., if the player is covered by the resource, a profit of $p_i \in \mathbb{R}_+$ is obtained. To analyse this setting, we define a *maximal covering location (MCL) situation* as a tuple $(N, L, p, r, d, \mathscr{D})$ with N, L, $p = (p_i)_{i\in N}$, $r = (r_i)_{i \in N}$, $d = (d_{ij})_{i\in N, j \in L}$ and $\mathscr{D}$ as described above. For short, we will use θ to refer to such an MCL situation $\theta = (N, L, p, r, d, \mathscr{D})$ and Θ for the set of MCL situations. In addition, for all $\theta \in \Theta$, we define $N_j = \{i \in N | d_{ij} \leq \mathscr{D}\}$ for all $j \in L$, $L_i = \{j \in L | d_{ij} \leq \mathscr{D}\}$ for all $i \in N$, and construct a corresponding (bipartite) graph $\mathscr{G} = (N, L, E)$ with N and L the sets of nodes and $E = \{(i,j)_{i \in N, j \in L_i}\}$ the set of edges. Note that an edge between player $i \in N$ and resource location $j \in L$ indicates that the distance between these nodes is no more than $\mathscr{D}$, implying that player i is covered when a resource is positioned at this location j.

3.2 Maximal covering location game

As some players may not own a resource, additional profit can be realized when resources are pooled amongst the players. In line with Church and ReVelle [1], we assume for any coalition $S \subseteq N$ that coverage of any player $i \in S$ by one (or possibly multiple) resource(s) results into a profit of p_i. As a consequence, any $S \subseteq N$ faces the joint problem of where to position the resources such that the sum of the individual profits (of coalition S) is maximized. For every MCL situation $\theta \in \Theta$ and all $S \subseteq N$ this corresponding MCL problem can be formulated as

$$
\begin{aligned}
MCL^\theta(S): \quad \max \quad & \sum_{i \in S} p_i \cdot y_i \\
\text{s.t.} \quad & y_i - \sum_{j \in L_i} x_j \leq 0 && \forall i \in S \\
& \sum_{j \in L} x_j \leq \sum_{i \in S} r_i \\
& x_j \in \{0,1\} && \forall j \in L \\
& y_i \in \{0,1\} && \forall i \in S.
\end{aligned}
$$

The first constraint ensures that the profit of player $i \in S$ is obtained only if at least one resource of coalition S is positioned within distance $\mathscr{D}$. The second constraint ensures that the total number of resources used does not exceed the number of available resources of coalition S. The third and fourth constraint enforce integrality of the variables. Note that a solution of the MCL problem indicates at which resource locations a resource is positioned and which players obtain a profit. In particular, if a resource is positioned at resource location $j \in L$, then $x_j = 1$ and otherwise $x_j = 0$. Similarly, if player $i \in S$ obtains profit p_i, then $y_i = 1$, otherwise $y_i = 0$.

In the remainder of this paper, we denote for every MCL situation $\theta \in \Theta$ and all $S \subseteq N$ the optimal value of $MCL^{\theta}(S)$ by $\mathbf{opt}(MCL^{\theta}(S))$.

Example 1. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3\}, L = \{4,5\}$, $p = (1,2,3)$, $r = (1,0,0)$, $d_{14} = d_{24} = d_{25} = d_{35} = 1$, $d_{15} = d_{34} = 2$ and $\mathscr{D} = 1$. Observe that $L_1 = \{4\}, L_2 = \{4,5\}$, and $L_3 = \{5\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{(1,4),(2,4),(2,5),(3,5)\}$ is represented in Figure 1.*

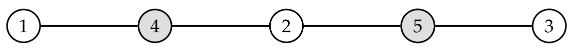

Figure 1: Graph corresponding to MCL situation

For coalition $S = \{1,3\}$, the maximization problem boils down to a trade off between a profit of 1, when the resource is positioned at location 4 and a profit of 3 when the resource is positioned at location 5. Hence, $\mathbf{opt}(MCL^{\theta}(\{1,3\})) = 3$. $\diamond$

We proceed with associating an MCL game to any MCL situation.

Definition 1. *For every MCL situation $\theta \in \Theta$, we call the game (N, v^{θ}) with*

$$v^{\theta}(S) = \begin{cases} 0 & \text{if} \quad S = \emptyset; \\ \mathbf{opt}(MCL^{\theta}(S)) & \text{if} \quad S \subseteq N, S \neq \emptyset, \end{cases}$$

the associated MCL game.

Now, we present an example of an MCL game.

Example 2. *Consider the situation of Example 1. The associated MCL game is presented in Table 1.* $\diamond$

Table 1: Corresponding profit per coalition of MCL game

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{\theta}(S)$	0	1	0	0	3	3	0	5

3.3 Properties of maximal covering location games

In this section, we present general properties of MCL games. We start by showing that the value of the union of two disjoint coalitions is larger than or equal to the sum of the values of the disjoint subcoalitions and that the value of every coalition is at least the value of any of its subcoalitions.

Proposition 1. *Every MCL game is superadditive and monotonic.*

The proof of Proposition 1 is relegated to the Appendix. We describe the intuition behind the proof. For superadditivity, optimal solutions of two disjoint coalitions are combined into a feasible solution for their union. For monotonicity, we use superadditivity in combination with non-negativity of the coalitional values.

A natural next step is to investigate whether the cores of MCL games are non-empty. The following example illustrates that this is not the case in general.

Example 3. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3,4\}, L = \{5,6,7,8\}, p = (1,1,1,1)$, $r = (1,1,0,0)$, $d_{15} = d_{17} = d_{25} = d_{26} = d_{36} = d_{37} = d_{48} = 1$, $d_{16} = d_{18} = d_{27} = d_{28} = d_{35} = d_{38} = d_{45} = d_{46} = d_{47} = 2$ and $\mathscr{D} = 1$. Then $L_1 = \{5,7\}$, $L_2 = \{5,6\}$, $L_3 = \{6,7\}$ and $L_4 = \{8\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{(1,5),(1,7),(2,5),(2,6),(3,6),(3,7),(4,8)\}$ is represented in Figure 2.*

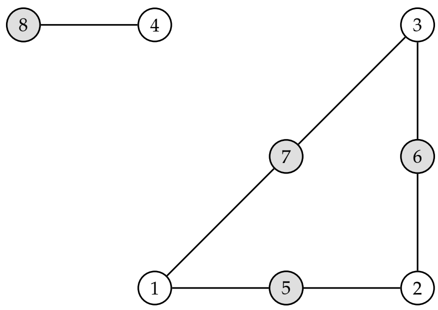

Figure 2: Graph corresponding to MCL situation

Now, observe that $v^\theta(N) = 3$, $v^\theta(N\backslash\{i\}) = 2$ for $i \in \{1,2\}$, and $v^\theta(N\backslash\{i\}) = 3$ for $i \in \{3,4\}$. Suppose the core is non-empty. Let $x \in C(N, v^\theta)$. As $x_i = \sum_{j \in N} x_j - \sum_{j \in N\backslash\{i\}} x_j = v^\theta(N) - \sum_{j \in N\backslash\{i\}} x_j \leq v^\theta(N) - v^\theta(N\backslash\{i\})$ for all $i \in N$, we obtain $x_1 \leq 1, x_2 \leq 1, x_3 \leq 0$, and $x_4 \leq 0$. This conflicts with efficiency, i.e., $\sum_{i \in N} x_i \leq 2 < 3 = v^\theta(N)$. Hence, we conclude that the core is empty. ◇

The graph corresponding to the MCL situation of Example 3 contains a cycle. In some other cooperative games related to problems in combinatorial optimization non-emptiness of the core is guaranteed when cycles are not present in the corresponding graph (Deng et al. [2]). For instance, for cost covering games this has been studied by Tamir [7]. One may wonder whether this holds for MCL games as well. The following 6-person MCL situation with a corresponding graph without cycles illustrates that this is not the case in general.

Example 4. *Let $\theta \in \Theta$ be an MCL situation with $N = \{1,2,3,4,5,6\}$, $L = \{7,8,9,10\}$, $p = (1,1,1,1,1,1)$, $r = (1,0,0,0,0,1)$, $d_{i7} = 1$ for $i \in \{1,2\}$, $d_{i7} = 2$ for $i \in \{3,4,5,6\}$, $d_{i8} = 1$ for $i \in \{2,3,5\}$, $d_{i8} = 2$ for $i \in \{1,4,6\}$, $d_{i9} = 1$ for $i \in \{3,4\}$, $d_{i9} = 2$ for $i \in \{1,2,5,6\}$, $d_{i10} = 1$ for $i \in \{5,6\}$, $d_{i10} = 2$ for $i \in \{1,2,3,4\}$, and $\mathscr{D} = 1$. Then, $L_1 = \{7\}, L_2 = \{7,8\}$, $L_3 = \{8,9\}$, $L_4 = \{9\}$, $L_5 = \{8,10\}$ and $L_6 = \{10\}$. The corresponding graph $\mathscr{G} = (N, L, E)$ with $E = \{(1,7),(2,7),(2,8),(3,8),(3,9), (4,9),(5,8),(5,10),(6,10)\}$ is represented in Figure 3.*

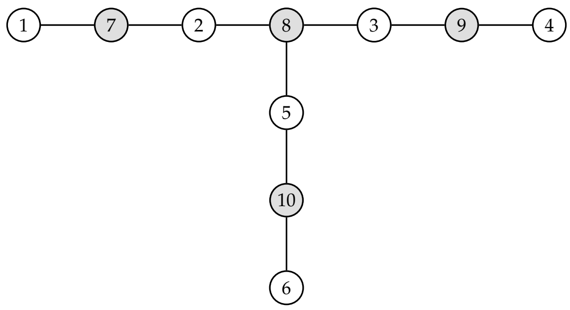

Figure 3: Graph corresponding to MCL situation

Now, observe that $v^\theta(N) = 4$ by positioning the two resources at any two resource locations. In addition, $v^\theta(N\backslash\{i\}) = 4$ for $i \in \{2,3,4,5\}$, and $v^\theta(N\backslash\{i\}) = 3$ for $i \in \{1,6\}$. Suppose the core is non-empty. Let $x \in C(N, v^\theta)$. As $x_i \leq v^\theta(N) - v^\theta(N\backslash\{i\})$ for all $i \in N$, we obtain $x_1 \leq 1, x_6 \leq 1$ and $x_i \leq 0$ for $i \in N\backslash\{1,6\}$. This conflicts with efficiency, i.e., $\sum_{i\in N} x_i \leq 2 < 4 = v^\theta(N)$. Hence, the core is empty. $\diamond$

Example 3 demonstrates that the core may be empty from 4-person games on. In addition, Example 4 shows that under the assumption that the corresponding graph contains no cycles the core may be empty from 6-person games on. With all this in mind we address the following two questions in the remainder of this paper; (i) is the core non-empty up to 3-person games in general and (ii) is the core non-empty up to 5-person games when the corresponding graph (of the MCL situation) contains no cycles? By addressing those issues, some other interesting results come along.

Proposition 2. *For every MCL situation $\theta \in \Theta$ and associated MCL game (N, v^θ), the core is non-empty if k players with $k \in \{0, 1, n-1, n\}$ own a resource.*

Proof : See Appendix.

Remark 1. If the condition of Proposition 2 is not satisfied, core non-emptiness is not guaranteed. In Example 3 with four players and two resources the core is empty.

As a direct consequence of Proposition 2, we can conclude that the core of every k-person MCL game with $k \in \{1, 2, 3\}$ is non-empty.

Theorem 1. *Every k-person MCL game with $k \in \{1, 2, 3\}$ has a non-empty core.*

We continue by addressing our second question of interest. For this, we introduce some definitions and present a proposition and two lemmas which are of interest by themselves as well. For every MCL situation $\theta \in \Theta$ and all $S \subseteq N$ we define $RMCL^\theta(S)$ as a relaxation of $MCL^\theta(S)$ where $x_j \geq 0$ for all $j \in L$ and $0 \leq y_i \leq 1$ for all $i \in S$. Note that $x_j \leq 1$ for all $j \in L$ is not taken into consideration. Based on this relaxation, we formulate a sufficient condition for non-emptiness of the core.

Proposition 3. *For any $\theta \in \Theta$ it holds that if* $\mathbf{opt}(RMCL^\theta(N)) = \mathbf{opt}(MCL^\theta(N))$, *the core of the associated MCL game (N, v^θ) is non-empty.*

Proof : See Appendix.

Remark 2. If the sufficient condition of Proposition 3 is not satisfied, core non-emptiness is not guaranteed. For instance, in Example 4 it holds that $\mathbf{opt}(MCL^\theta(N)) = 4$, $\mathbf{opt}(RMCL^\theta(N)) = 4.5$ (resulting from $x_j = \frac{1}{2}$ for all $j \in L$), and $C(N, v^\theta) = \emptyset$. In addition, the condition of Proposition 3 is not necessary. For instance, in Example 4 with $r_2 = r_3 = 1$ and $r_i = 0$ for all $i \in N \backslash \{2, 3\}$, it holds that $C(N, v^\theta) = \{(0, 2, 2, 0, 0, 0, 0)\} \neq \emptyset$.

A *square submatrix* of a matrix $A \in \mathbb{R}^{w \times z}$ (where w is the number of rows and z is the number of columns) is a matrix $A' \in \mathbb{R}^{q \times q}$ formed by selecting q rows and q columns from the matrix A. Moreover, a matrix is *totally unimodular* if every square submatrix of A has determinant equal to $+1, -1$ or 0.

Lemma 1. *Let A be a totally unimodular $w \times z$ matrix and let $b \in \mathbb{Z}^w$ and $c \in \mathbb{R}^z$. Then the linear programming problem*

$$\max\{cx | x \geq 0, Ax \leq b\}$$

has integer optimal solutions, whenever it has a finite optimum.

Proof : See e.g. Wolsey [8, p.40].

For all $\theta = (N, L, p, r, d, \mathscr{D}) \in \Theta$ and all $j \in L$, we define $\theta^{-j} = (N, L^{-j}, p, r, d^{-j}, \mathscr{D})$, where $L^{-j} = L \backslash \{j\}$ and $d^{-j} = (d_{ij})_{i \in N, j \in L \backslash \{j\}}$. In addition, for every $\theta \in \Theta$, resource location $j \in L$ is called *obsolete* if there exists $k \in L \backslash \{j\}$ for which $N_j \subseteq N_k$.

Lemma 2. *For any $\theta \in \Theta$ with an obsolete resource location $j \in L$, it holds that*

$$v^{\theta}(M) = v^{\theta^{-j}}(M) \text{ for all } M \subseteq N.$$

Note that the result of Lemma 2 follows directly, as (for every $M \subseteq N$) there exists a $k \in L \backslash \{j\}$ with $N_j \subseteq N_k$ which makes k superfluous. Now, we are able to affirmatively answer our second question of interest. We provide a sketch of the proof here. The complete proof is relegated to the appendix.

As for every MCL situation with $0, 1, |N| - 1$ and $|N|$ resources the core of the associated MCL game is non-empty (Proposition 2), it suffices to focus on the following MCL situations only; (i) $|N| = 4$ with two resources, (ii) $|N| = 5$ with two resources, and (iii) $|N| = 5$ with three resources. For all cases, it holds that if the number of resource locations is less than or equal to the number of resources, the core is non-empty. This holds as allocating p_i to all players $i \in N$ with $L_i \neq \emptyset$, i.e., for which there exists a resource location within distance $\mathscr{D}$, results into a core element. In addition, for every MCL situation with an obsolete resource location there exists another MCL situation without this obsolete resource location such that the coalitional values of the MCL games resulting from those MCL situations coincide (Lemma 2). Hence, if the core is non-empty for the game corresponding to MCL situation without the obsolete resource location, it is the case for the other game as well. So, it suffices to consider the three cases, including that (i) $|L| \geq 3$ and (ii) the corresponding graphs are free of obsolete resource locations and cycles. In Figure 4, the possible graphs for $|N| = 4$ with $|L| \geq 3$ are presented.

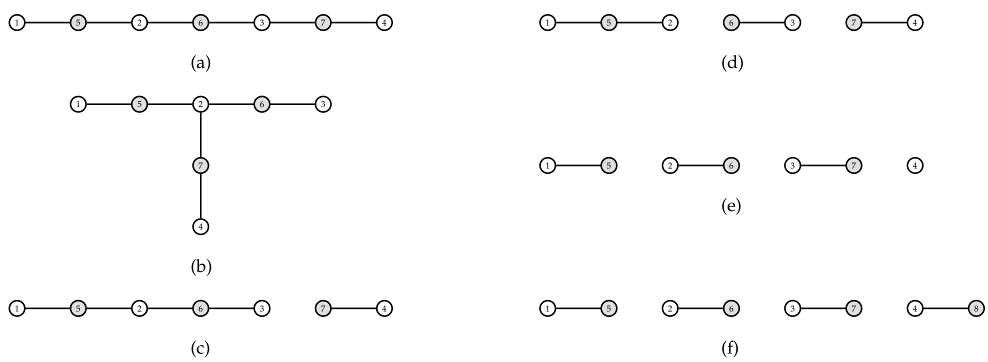

Figure 4: Possible graphs with $|N| = 4$ and $|L| \geq 3$.

Note that duplicates of the graphs of Figure 4 due to relabeling of the player set and resource set are removed. Similarly, one can present the possible graphs (without obsolete resource locations and cycles) for case $|N| = 5$ with $|L| \geq 3$. These graphs are presented in Figure 5. Note that duplicates of the graphs of Figure 5 due to relabeling of the player set and resource set are removed (again).

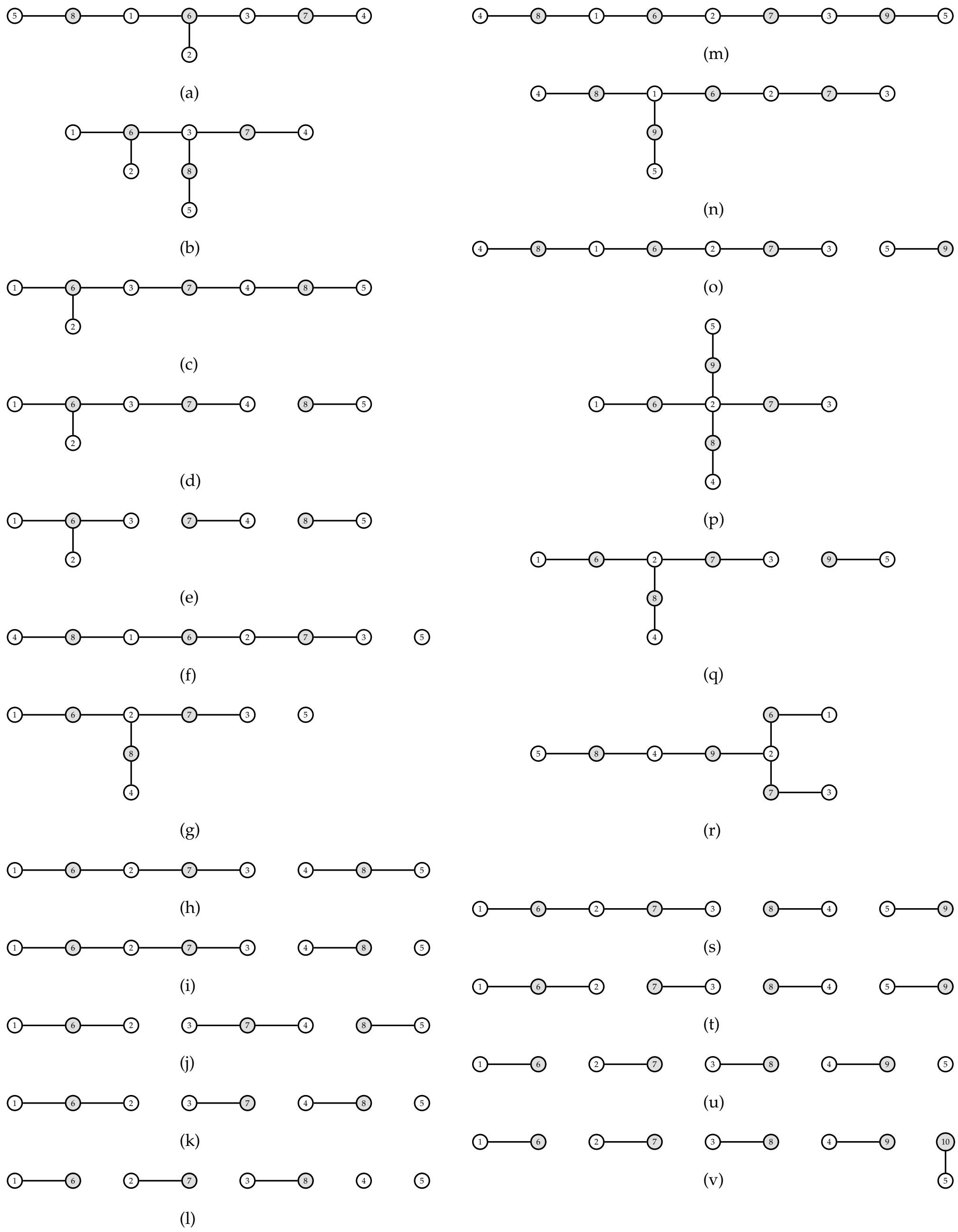

Figure 5: Possible graphs with $|N| = 5$ and $|L| \geq 3$

For all remaining MCL situations, which *all* have a corresponding graph as presented in Figure 4 or Figure 5 (possibly after some relabeling of the player set and the resource location set), we can (re)formulate the relaxation of the MCL problem in standard LP-form, i.e., in matrix form $Ax \leq b$ with $x \geq 0$.

Example 5. *Let $\theta \in \Theta$ be an MCL situation with corresponding graph (e) of Figure 4. Then, for the relaxation of the MCL problem, we obtain the following A and b.* $\diamond$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

For all these MCL situations, we can show that matrix A is totally unimodular[1] and vector b (of the standard LP-form) has all integer entries. In addition, it holds that $p_i \in \mathbb{R}_+$ for all $i \in N$ and so, we can conclude that the relaxation of the MCL problem has an integer optimal solution (Lemma 1). Hence, the optimal value of the MCL problem coincides with the optimal value of the relaxation of the MCL problem and thus, the core is non-empty (Proposition 3). This leads to the our final theorem.

Theorem 2. *Every k-person MCL game with $k \in \{1, 2, 3, 4, 5\}$ and no cycles in the corresponding graph has a non-empty core.*

Remark 3. The more general case in which each player may own *several* resources to cover *multiple* demand points can be deducted from our situation easily by merging several players (with and without a resource) into one (super)player. All sufficient conditions can be translated to this situation easily.

[1]For all 28 possible graphs (of Figure 4 and Figure 5 together) one can check via standard software packages that every submatrix of A has determinant 1,0 or -1.

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Appendix

Proof of Proposition 1

First we show that every MCL game is superadditive. Let $\theta \in \Theta$ be an MCL situation and (N, v^θ) be the associated MCL game. In addition, let $S, T \subseteq N$ with $S \cap T = \emptyset$. Moreover, let $((x_j^S)_{j \in L}, (y_i^S)_{i \in S})$ be an optimal solution of coalition S, and $((x_j^T)_{j \in L}, (y_i^T)_{i \in T})$ be an optimal solution of coalition T. Now, we construct a solution $((x_j)_{j \in L}, (y_i)_{i \in S \cup T})$, given by

$$x_j = \max\{x_j^S, x_j^T\} \quad \forall j \in L, \quad y_i = \begin{cases} y_i^S & \text{if } i \in S \\ \\ y_i^T & \text{if } i \in T. \end{cases}$$

We claim that $((x_j)_{j\in L}, (y_i)_{i\in S\cup T})$ is feasible. Let $i \in S$, then

$$y_i - \sum_{j\in L_i} x_j = y_i^S - \sum_{j\in L_i} \max\{x_j^S, x_j^T\} \leq y_i^S - \sum_{j\in L_i} x_j^S \leq 0,$$

where the equality holds by definition. The first inequality holds as $x_j^S \leq \max\{x_j^S, x_j^T\}$ for all $j \in L_i$, and the second inequality holds by the feasibility of $((x_j^S)_{j\in L}, (y_i^S)_{i\in S})$. In a similar way, this holds for any $i \in T$. Now, observe that

$$\begin{aligned}\sum_{j\in L} x_j - \sum_{i\in S\cup T} r_i &= \sum_{j\in L} \max\{x_j^S, x_j^T\} - \sum_{i\in S\cup T} r_i \\ &\leq \sum_{j\in L} x_j^S + \sum_{j\in L} x_j^T - \sum_{i\in S\cup T} r_i \\ &= \sum_{j\in L} x_j^S - \sum_{i\in T} r_i + \sum_{j\in L} x_j^T - \sum_{i\in S} r_i \leq 0,\end{aligned}$$

where the first and second equality hold by definition. The first inequality holds as $\max\{x_j^S, x_j^T\} \leq x_j^S + x_j^T$ for all $j \in L$, and the second inequality holds by the feasibility of $((x_j^S)_{j\in L}, (y_i^S)_{i\in S})$ and $((x_j^T)_{j\in L}, (y_i^T)_{i\in T})$. Finally, observe that $x_j \in \{0,1\}$ for all $j \in L$, and $y_i \in \{0,1\}$ for all $i \in S \cup T$. Hence, $((x_j)_{j\in L}, (y_i)_{i\in S\cup T})$ is a feasible solution. From this, we conclude that

$$v^\theta(S\cup T) \geq \sum_{i\in S\cup T} p_i \cdot y_i = \sum_{i\in S} p_i \cdot y_i^S + \sum_{i\in T} p_i \cdot y_i^T = v^\theta(S) + v^\theta(T),$$

where the first equality holds as $((x_j)_{j\in L}, (y_i)_{i\in S\cup T})$ is a feasible solution, with associated profit that is at most as much as the profit under an optimal solution of coalition $S \cup T$. The first and second equality hold by definition.

Now, we show that every MCL game is monotonic. Let $\theta \in \Theta$ be an MCL situation and (N, v^θ) be the associated MCL game. In addition, let $S, T \subseteq N$ with $S \subseteq T$. First, observe that $v(T\backslash S) \geq 0$ as $\sum_{i\in T\backslash S} p_i \cdot y_i = 0$ under feasible solution $x_j = 0$ for all $j \in L$ and $y_i = 0$ for all $i \in T\backslash S$. Then, observe that

$$v^\theta(S) \leq v^\theta(S) + v^\theta(T\backslash S) \leq v^\theta(T),$$

where the first equality holds as $v^\theta(T\backslash S) \geq 0$. The second equality holds as MCL games are superadditive. This concludes the proof. $\square$

Proof of Proposition 2

We consider the 4 possibilities of k separately.

1. Let $\theta_0 \in \Theta$ be an MCL situation where no player owns a resource, i.e., $k = 0$, and (N, v^{θ_0}) be the associated MCL game. As no player owns a resource, it holds that $v^{\theta_0}(S) = 0$ for all $S \subseteq N$. Hence, the game is additive and from this we conclude that the core is non-empty.

2. Let $\theta_1 \in \Theta$ be an MCL situation where exactly one player owns a resource, i.e., $k = 1$, and (N, v^{θ_1}) be the associated MCL game. Assume that player $i^* \in N$ owns a resource. Now, let $x_{i^*} = v^{\theta_1}(N)$ and $x_i = 0$ for all $i \in N \backslash \{i^*\}$. Then, for any $S \subseteq N$ with $i^* \notin S$, it holds that $\sum_{i \in S} x_i = 0 = v^{\theta_1}(S)$. For any $S \subseteq N$ with $i^* \in S$, it holds that $\sum_{i \in S} x_i = x_{i^*} + 0 = v^{\theta_1}(N) \geq v^{\theta_1}(S)$. The last inequality holds as MCL games are monotonic. Finally, observe that $\sum_{i \in N} x_i = v^{\theta_1}(N)$ and thus $x \in C(N, v^{\theta_1})$.

3. Let $\theta_{n-1} \in \Theta$ be an MCL situation where everyone owns a resource, except for one player, i.e., $k = n - 1$, and $(N, v^{\theta_{n-1}})$ be the associated MCL game. Let $i^* \in N$ be the player with $r_{i^*} = 0$. If $|N| \leq 2$, we end up in case 1 or 2. Hence, we can restrict attention to $|N| > 2$. We distinguish between two cases (and two subcases per case).

Case a. For all $i \in N$ it holds that $L_i \neq \emptyset$.

Case a.1 For all $i, i' \in N$ with $i \neq i'$ it holds that $L_i \cap L_{i'} = \emptyset$.

Observe that for every $i \in N$ there exists a $j \in L_i$ such that $j \notin L_{i'}$ for all $i' \in N \backslash \{i\}$. As $n - 1$ resources are available only, it follows that $v^{\theta_{n-1}}(N) = \sum_{i \in N} p_i - p_{\min(N)}$, where $p_{\min(S)} = \min_{i \in S} p_i$ for all $S \subseteq N$. Let $x_{i^*} = p_{i^*} - p_{\min(N)}$ and $x_i = p_i$ for all $i \in N \backslash \{i^*\}$. Note that $p_{\min(N)} \leq p_{\min(S)}$ for all $S \subseteq N$. Now, for any $S \subseteq N$ with $i^* \notin S$, it holds that $\sum_{i \in S} x_i = \sum_{i \in S} p_i = v^{\theta_{n-1}}(S)$, where the last equality holds as every $i \in S$ owns a resource. In addition, for any $S \subseteq N$ with $i^* \in S$, it holds that $\sum_{i \in S} x_i = \sum_{i \in S} p_i - p_{\min(N)} \geq \sum_{i \in S} p_i - p_{\min(S)} = v^{\theta_{n-1}}(S)$. Finally, observe that $\sum_{i \in N} x_i = \sum_{i \in N} p_i - p_{\min(N)} = v^{\theta_{n-1}}(N)$. We conclude that $x \in C(N, v^{\theta})$.

Case a.2 There exist $i, i' \in N$ with $i \neq i'$ for which $L_i \cap L_{i'} \neq \emptyset$.

Observe that $|L_i| \geq 1$ for all $i \in N$ and that there exists a $j \in L$ and $k, l \in N$ for which $j \in L_k, L_l$. Hence, $v^{\theta_{n-1}}(N) = \sum_{i \in N} p_i$. Then, let $x_i = p_i$ for all $i \in N$. Observe that for any $S \subseteq N$, it holds that $\sum_{i \in S} x_i = \sum_{i \in S} p_i \geq v^{\theta_{n-1}}(S)$ and $\sum_{i \in N} x_i = \sum_{i \in N} p_i = v^{\theta_{n-1}}(N)$. We conclude that $x \in C(N, v^{\theta})$.

Case b. There exists an $i \in N$ for which $L_i = \emptyset$.

Case b.1 $L_{i^*} = \emptyset$.

As player i^* (also) has no resource, it can add nothing to any coalition. Hence, we can restrict ourselves to a subgame without player i^*, i.e., with player set $N \backslash \{i^*\}$ where each player owns a resource. Then, we end up in case 4.

Case b.2 $L_{i^*} \neq \emptyset$.

Let $T = \{t \in N | L_t = \emptyset\}$. As $L_{i^*} \neq \emptyset$, it holds that $1 \leq |T| \leq n-1$. Then, in the grand coalition, for any $b \in B = N \backslash T$ there is a resource available and so $v^{\theta_{n-1}}(N) = \sum_{i \in B} p_i$. Let $x_i = p_i$ for all $i \in B$. As for all players $t \in T$ it holds that they can never add any profit, it holds for any $S \subseteq N$ that $\sum_{i \in S} x_i \geq \sum_{i \in S \cap B} x_i = v^{\theta_{n-1}}(S)$. Finally, $\sum_{i \in N} x_i = \sum_{i \in N \cap B} p_i = v^{\theta_{n-1}}(N)$. Hence, $x \in C(N, v^{\theta})$.

4. Let $\theta_n \in \Theta$ be an MCL situation where every player owns a resource, i.e., $k = n$, and (N, v^{θ_n}) be the associated MCL game. Let $H = \{h \in N | L_h \neq \emptyset\}$. As a consequence, $v^{\theta_n}(S) = \sum_{i \in S \cap H} p_i$ for all $S \subseteq N$. Now, let $x_i = p_i$ for each $i \in H$ and $x_i = 0$ for every $i \in N \backslash H$. Then, observe that $\sum_{i \in S} x_i = \sum_{i \in S \cap H} p_i = v^{\theta_n}(S)$ for each $S \subseteq N$. We conclude that the core is non-empty. $\square$

Proof of Proposition 3

Let $\theta \in \Theta$ and assume that $\mathbf{opt}(MCL^{\theta}(N)) = \mathbf{opt}(RMCL^{\theta}(N))$. Let $S \subseteq N$. Note that $RMCL^{\theta}(S)$ is feasible (as $x_j = 0$ for all $j \in L$ and $y_i = 0$ for all $i \in S$ is a feasible solution) and bounded (by $\sum_{i \in S} p_i + 1$). Then, for the dual of $RMCL^{\theta}(S)$, which is given by

$$
\begin{aligned}
DRMCL^{\theta}(S): \ \min \quad & b \cdot \sum_{i \in S} r_i + \sum_{i \in S} c_i \\
s.t. \quad & a_i + c_i \geq p_i \quad \forall i \in S \\
& -\sum_{i \in N_j^S} a_i + b \geq 0 \quad \forall j \in L \\
& a_i, c_i \geq 0 \quad \forall i \in S \\
& b \geq 0
\end{aligned}
$$

where $N_j^S = \{i \in S | d_{ij} \leq \mathscr{D}\}$, it holds based on the duality theorem of linear programming (see e.g. Schrijver [6, p.90]) that

$$
\mathbf{opt}(DRMCL^{\theta}(S)) = \mathbf{opt}(RMCL^{\theta}(S)). \tag{1}
$$

Based on our assumption, it holds for $S = N$ that

$$\mathbf{opt}(MCL^\theta(N)) = \mathbf{opt}(DRMCL^\theta(N)). \tag{2}$$

Again, let $S \subseteq N$. We claim that the restriction of any feasible solution $((a_i)_{i\in N}, b, (c_i)_{i\in N})$ of $DRMCL^\theta(N)$ to S defined by $((a'_i)_{i\in S}, b', (c'_i)_{i\in S})$ with

$$\begin{aligned}
&a'_i = a_i \text{ for all } i \in S\\
&b' = b\\
&c'_i = c_i \text{ for all } i \in S
\end{aligned}$$

is a feasible solution of $DRMCL^\theta(S)$. Let $i \in S$. Then $p_i \leq a_i + c_i = a'_i + c'_i$. Again, let $i \in S$. Then $0 \leq -\sum_{i\in N_j^N} a_i + b \leq -\sum_{i\in N_j^S} a_i + b = -\sum_{i\in N_j^S} a'_i + b'$. Finally, as $a'_i = a_i \geq 0$ for all $i \in S$, $b' = b \geq 0$, and $c'_i = c_i \geq 0$ for all $i \in S$, we conclude that $((a'_i)_{i\in S}, b', (c'_i)_{i\in S})$ is a feasible solution of $DRMCL^\theta(S)$.

Now, let $((a^*_i)_{i\in N}, b^*, (c^*_i)_{i\in N})$ be an optimal solution of dual problem $DRMCL^\theta(N)$. Note that this solution exists by the duality theorem of linear programming (again). We construct a payoff vector $z = (z_i)_{i\in N}$ as follows

$$z_i = b^* \cdot r_i + c^*_i \text{ for all } i \in N.$$

Now, observe that

$$\begin{aligned}
\sum_{i\in N} z_i = \sum_{i\in N} [b^* \cdot r_i + c^*_i] = b^* \sum_{i\in N} r_i + \sum_{i\in N} c^*_i\\
&= \mathbf{opt}(DRMCL^\theta(N))\\
&= \mathbf{opt}(MCL^\theta(N))\\
&= v^\theta(N).
\end{aligned}$$

where the third equality holds by equation (2). In addition, let $S \subseteq N$ and $((a'^*_i)_{i\in S}, b'^*, (c'^*_i)_{i\in S})$ be the restriction of $((a^*_i)_{i\in N}, b^*, (c^*_i)_{i\in N})$ to coalition S. Then, observe that

$$\begin{aligned}
\sum_{i\in S} z_i = \sum_{i\in S} [b^* \cdot r_i + c^*_i] = b'^* \sum_{i\in S} r_i + \sum_{i\in S} c'^*_i &\geq \mathbf{opt}(DRMCL^\theta(S))\\
&= \mathbf{opt}(RMCL^\theta(S))\\
&\geq \mathbf{opt}(MCL^\theta(S))\\
&= v^\theta(S),
\end{aligned}$$

where the first inequality holds as the value of feasible solution $((a'^*_i)_{i\in S}, b'^*, (c'^*_i)_{i\in S})$ is more than or equal to the optimal value of $DRMCL^\theta(S)$. The first equality holds by equation (1). The second inequality holds as we restrict to integer solutions only. As there exists a $z = (z_i)_{i \in N}$ for which $\sum_{i \in N} z_i = v^\theta(N)$ and $\sum_{i \in S} z_i \geq v^\theta(S)$ for all $S \subseteq N$, we conclude that the core of (N, v^θ) is non-empty. $\square$

In order to prove Theorem 2 we introduce some new terminology. Graph $\mathscr{G} = (N, L, E)$ is *isomorphic* to graph $\widetilde{\mathscr{G}} = (\widetilde{N}, \widetilde{L}, \widetilde{E})$ if there is a one-to-one correspondence between the nodes in N and $\widetilde{N}$ and a one-to-one correspondence between the nodes in L and $\widetilde{L}$ such that a node in N and a node in L are connected directly if and only if the corresponding nodes in $\widetilde{N}$ and $\widetilde{L}$ are connected directly. In addition, we present two useful lemmas.

Lemma 3. *Let $\theta \in \Theta$ with $|N| = 4$ and $|L| \geq 3$. If the corresponding graph $\mathscr{G} = (N, L, E)$ has no cycles nor obsolete resource locations, then graph $\mathscr{G} = (N, L, E)$ is isomorphic to one of the graphs represented in Figure 4.*

Proof of Lemma 3

If there exists a $j \in L$ for which $N_j = \emptyset$ then this resource location is obsolete. So, from now on, suppose $1 \leq |N_j| \leq 4$ for all $j \in L$. We distinguish between cases $|L| = 3, |L| = 4$ and $|L| \geq 5$. In each case, we force a contradiction (by showing that the graph has an obsolete resource location or a cycle) or present the possible graph(s).

Case 1. $|L| = 3$.

Let $L = \{5, 6, 7\}$ with $|N_5| \geq |N_6| \geq |N_7|$ without loss of generality. Now, we condition on the cardinality of N_5, N_6 and N_7.

Case 1.a. $|N_5| = 4$.

Note that $6 \in L$ is obsolete if $|N_6| = 1$. If $|N_6| \geq 2$, a cycle is formed.

Case 1.b. $|N_5| = 3, |N_6| = 3$.

We have $|N_5 \cap N_6| \geq 2$. A cycle is formed.

Case 1.c. $|N_5| = 3, |N_6| = 2$.

In order to avoid cycles, we have $|N_5 \cap N_6| = 1$. Let $N_5 = \{1, 2, 3\}$ and $N_6 = \{2, 4\}$ without loss of generality. If $|N_7| = 1$ then $7 \in L$ is obsolete. If $|N_7| = 2$ a cycle is formed.

Case 1.d. $|N_5| = 3, |N_6| = 1$.

In order to avoid an obsolete resource location, we have $N_5 \cap N_6 = \emptyset$. Let $N_5 = \{1, 2, 3\}$ and $N_6 = \{4\}$ without loss of generality. If $|N_7| = 1$ then $7 \in L$ is obsolete.

Case 1.e. $|N_5| = 2, |N_6| = 2$.

In order to avoid cycles, we have $|N_5 \cap N_6| \leq 1$. Let (i) $N_5 = \{1,2\}$ and $N_6 = \{2,3\}$ or (ii) $N_5 = \{1,2\}$ and $N_6 = \{3,4\}$ without loss of generality.

(i) If $N_7 \in \{\{1,2\},\{1,3\},\{2,3\}\}$ a cycle is formed. For $N_7 \in \{\{1,4\},\{3,4\}\}$ we obtain a graph isomorphic to graph (a) (of Figure 4). For $N_7 = \{2,4\}$ we obtain a graph isomorphic to graph (b). If $N_7 \in \{\{1\},\{2\},\{3\}\}$ we obtain an obsolete resource location. For $N_7 = \{4\}$ we obtain a graph isomorphic to graph (c).

(ii) If $N_7 \in \{\{1,2\},\{3,4\}\}$ a cycle is formed. For $N_7 \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ we obtain a graph isomorphic to graph (a). If $|N_7| = 1$, then resource location $7 \in L$ is obsolete.

Case 1.f. $|N_5| = 2, |N_6| = 1, |N_7| = 1$.

In order to avoid obsolete resource locations, we have $N_5 \cap N_6 = N_5 \cap N_7 = N_6 \cap N_7 = \emptyset$. Let $N_5 = \{1,2\}, N_6 = \{3\}$, and $N_7 = \{4\}$ without loss of generality. Then, we obtain a graph isomorphic to graph (d).

Case 1.g. $|N_5| = 1, |N_6| = 1, |N_7| = 1$.

In order to avoid obsolete resource locations, we have $N_5 \cap N_6 = N_5 \cap N_7 = N_6 \cap N_7 = \emptyset$. Let $N_5 = \{1\}, N_6 = \{2\}$, and $N_7 = \{3\}$ without loss of generality. Then, we obtain a graph isomorphic to graph (e).

Case 2. $|L| = 4$.

Let $L = \{5,6,7,8\}$ with $|N_5| \geq |N_6| \geq |N_7| \geq |N_8|$ without loss of generality. We condition on the cardinality of N_5, N_6, N_7 and N_8. We make use of the results of Case 1. For some cases this implies that we can conclude immediately that no graph exists (with no cycles nor obsolete resource locations). For the other cases, we can simply start with the obtained (sub)graphs and condition on N_8.

Case 2.a. $|N_5| \geq 3$.

The reasoning is identical to *cases 1.a, 1.b, 1.c, and 1.d.*

Case 2.b. $|N_5| = 2, |N_6| = 2$.

By using the (same) reasoning of *case 1.e* we end up with subgraphs $(a), (b)$ and (c).

(a) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{3,4\}$. If $|N_8| = 2$ a cycle is formed. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

(b) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{2,4\}$. If $|N_8| = 2$ a cycle is formed. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

(c) $N_5 = \{1,2\}, N_6 = \{2,3\}$, and $N_7 = \{4\}$. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

Case 2.c. $|N_5| = 2, |N_6| = 1, |N_7| = 1$.

By using the (same) reasoning of *Case 1.f* we end up with subgraph (d).

(d) $N_5 = \{1,2\}, N_6 = \{3\}$, and $N_7 = \{4\}$. If $|N_8| = 1$, resource location $8 \in L$ is obsolete.

Case 2.d. $|N_5| = 1, |N_6| = 1, |N_7| = 1$.

By using the (same) reasoning of *Case 1.g* we end up with subgraphs (e).

(e) $N_5 = \{1\}, N_6 = \{2\}$, and $N_7 = \{3\}$. If $N_8 \in \{\{1\},\{2\},\{3\}\}$ we obtain an obsolete resource location. For $N_8 = \{4\}$, we obtain a graph isomorphic to graph (f).

Case 3. $|L| \geq 5$.

Let $L = \{5,6,\ldots,k\}$ with $k \geq 9$ where $|N_5| \geq |N_6| \geq \ldots| \geq |N_k|$ without loss of generality. In line with our discussion at the start of Case 2, we can simply start with the obtained (sub)graphs and condition on N_9.

Case 3.a. $|N_5| \geq 2$.

The reasoning is identical to *cases 2.a, 2.b, and 2.c*.

Case 3.b. $|N_5| = 1, |N_6| = 1, |N_7| = 1$.

By using the (same) reasoning of *Case 2.d* we end up with subgraphs (e).

(f) $N_5 = \{1\}, N_6 = \{2\}, N_7 = \{3\}$, and $N_8 = \{4\}$. If $|N_9| = 1$, then resource location $9 \in L$ is obsolete.

We conclude that the graph (without cycles and obsolete resource locations) is isomorphic to one of the graphs of Figure 4. □

Lemma 4. *Let $\theta \in \Theta$ with $|N| = 5$ and $|L| \geq 3$. If the corresponding graph $\mathscr{G} = (N, L, E)$ has no cycles nor obsolete resource locations, then graph $\mathscr{G} = (N, L, E)$ is isomorphic to one of the graphs of Figure 5.*

Proof of Lemma 4

If there exists a $j \in L$ for which $N_j = \varnothing$ then this resource location is obsolete. So, from now on, suppose $1 \leq |N_j| \leq 5$ for all $j \in L$. We distinguish between cases $|L| = 3, |L| = 4$, $|L| = 5$ and $|L| \geq 6$. In each case, we force a contradiction (by showing that the graph has an obsolete resource location or a cycle) or present the possible graph(s).

Case 1. $|L| = 3$.

Let $L = \{6, 7, 8\}$ and assume without loss of generality that $|N_6| \geq |N_7| \geq |N_8|$. We condition on the cardinality of N_6, N_7 and N_8.

Case 1.a. $|N_6| = 5$.

Note that $7 \in L$ is obsolete if $|N_7| = 1$. If $|N_7| \geq 2$, a cycle is formed.

Case 1.b. $|N_6| = 4, |N_7| \geq 3$.

We have $|N_6 \cap N_7| \geq 2$. Hence, a cycle is formed.

Case 1.c. $|N_6| = 4, |N_7| = 2$.

In order to avoid cycles, we have $|N_6 \cap N_7| = 1$. Let $N_6 = \{1, 2, 3, 4\}$ and $N_7 = \{4, 5\}$ without loss of generality. If $|N_8| = 1$ resource location $8 \in L$ is obsolete. If $|N_8| = 2$ a cycle is formed.

Case 1.d. $|N_6| = 4, |N_7| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = \emptyset$. Let $N_6 = \{1, 2, 3, 4\}$ and $N_7 = \{5\}$ without loss of generality. If $|N_8| = 1$ then $8 \in L$ is obsolete.

Case 1.e. $|N_6| = 3, |N_7| = 3$.

In order to avoid cycles, we have $|N_6 \cap N_7| = 1$. Let $N_6 = \{1, 2, 3\}$ and $N_7 = \{3, 4, 5\}$ without loss of generality. If $2 \leq |N_8| \leq 3$ a cycle is formed. If $|N_8| = 1$ then $8 \in L$ is obsolete.

Case 1.f. $|N_6| = 3, |N_7| = 2, |N_8| = 2$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1, 2, 3\}$ and $N_7 = \{3, 4\}$ or (ii) $N_6 = \{1, 2, 3\}$ and $N_7 = \{4, 5\}$ without loss of generality.

(i) If $N_8 \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ a cycle is formed. For $N_8 \in \{\{1, 5\}, \{2, 5\}\}$ we obtain a graph isomorphic to (a) (of Figure 5). For $N_8 = \{3, 5\}$ we obtain a graph isomorphic to (b). For $N_8 = \{4, 5\}$ we obtain a graph isomorphic to graph (c).

(ii) If $N_8 \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}\}$ a cycle is formed. For $N_8 \in \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$ we obtain a graph isomorphic to graph (c).

Case 1.g. $|N_6| = 3, |N_7| = 2, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1,2,3\}$ and $N_7 = \{3,4\}$ or (ii) $N_6 = \{1,2,3\}$ and $N_7 = \{4,5\}$ without loss of generality.

(i) If $N_8 \in \{\{1\},\{2\},\{3\},\{4\}\}$ the resource location is obsolete. For $N_8 = \{5\}$, we obtain a graph isomorphic to graph (d).

(ii) If $|N_8| = 1$, $8 \in L$ is obsolete.

Case 1.h. $|N_6| = 3, |N_7| = 1, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1,2,3\}$, $N_7 = \{4\}$ and $N_8 = \{5\}$ without loss of generality. Then, we obtain a graph isomorphic to (e).

Case 1.i. $|N_6| = 2, |N_7| = 2, |N_8| = 2$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1,2\}$ and $N_7 = \{2,3\}$ or (ii) $N_6 = \{1,2\}$ and $N_7 = \{3,4\}$ without loss of generality.

(i) If $N_8 \in \{\{1,2\},\{1,3\},\{2,3\}\}$ a cycle is formed. For $N_8 \in \{\{1,4\},\{1,5\},\{3,4\},\{3,5\}\}$ we obtain a graph isomorphic to (f). For $N_8 \in \{\{2,4\},\{2,5\}\}$ we obtain a graph isomorphic to (g). For $N_8 = \{4,5\}$ we obtain a graph isomorphic to (h).

(ii) If $N_8 \in \{\{1,2\},\{3,4\}\}$ a cycle is formed. For $N_8 \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ we obtain a graph isomorphic to (f). For $N_8 \in \{\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to (h).

Case 1.j. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $|N_6 \cap N_7| \leq 1$. Let (i) $N_6 = \{1,2\}$ and $N_7 = \{2,3\}$ or (ii) $N_6 = \{1,2\}$ and $N_7 = \{3,4\}$ without loss of generality.

(i) If $N_8 \in \{\{1\},\{2\},\{3\}\}$ we obtain an obsolete resource location. For $N_6 \in \{\{4\},\{5\}\}$ we obtain a graph isomorphic to (i).

(ii) If $N_8 \in \{\{1\},\{2\},\{3\},\{4\}\}$ we obtain an obsolete resource location. For $N_8 = \{5\}$, we obtain a graph isomorphic to (j).

Case 1.k. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1,2\}$, $N_7 = \{3\}$ and $N_8 = \{4\}$ without loss of generality. Then, we obtain a graph isomorphic to (k).

Case 1.l. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

In order to avoid an obsolete resource location, we have $N_6 \cap N_7 = N_6 \cap N_8 = N_7 \cap N_8 = \emptyset$. Let $N_6 = \{1\}$, $N_7 = \{2\}$, and $N_8 = \{3\}$ without loss of generality. Then, we obtain a graph isomorphic to (l).

Case 2. $|L| = 4$.

Let $L = \{6,7,8,9\}$ with $|N_6| \geq |N_7| \geq |N_8| \geq |N_9|$ without loss of generality. In line with our discussion at the start of Case 2 of the proof of Lemma 3, we can simply start with the obtained (sub)graphs and condition on N_9.

Case 2.a. $|N_6| \geq 4$.

The reasoning is identical to *cases 1.a, 1.b, 1.c, and 1.d*.

Case 2.b. $|N_6| = 3, |N_7| = 3$.

The reasoning is identical to *case 1.e*.

Case 2.c. $|N_6| = 3, |N_7| = 2, |N_8| = 2$.

By using the (same) reasoning of *Case 1.f* we end up with subgraphs $(a), (b)$ and (c).

(a) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{1,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

(b) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{3,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

(c) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{4,5\}$. If $|N_9| = 2$ a cycle is formed. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.d. $|N_6| = 3, |N_7| = 2, |N_8| = 1$.

By using the (same) reasoning of *Case 1.g* we end up with subgraph (d).

(d) $N_6 = \{1,2,3\}, N_7 = \{3,4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.e. $|N_6| = 3, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 1.h* we end up with subgraph (e).

(e) $N_6 = \{1,2,3\}, N_7 = \{4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.f. $|N_6| = 2, |N_7| = 2, |N_8| = 2$.

By using the (same) reasoning of *Case 1.i* we end up with subgraph $(f), (g)$ and (h).

(f) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ a cycle is formed. For $N_9 \in \{\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to

(m). For $N_9 \in \{\{1,5\},\{2,5\}\}$ we obtain a graph isomorphic to (n). If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (o).

(g) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ a cycle is formed. For $N_9 \in \{\{1,5\},\{3,5\},\{4,5\}\}$ we obtain a graph isomorphic to (n). For $N_9 = \{2,5\}$ we obtain a graph isomorphic to (p). If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (q).

(h) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4,5\}$. If $N_9 \in \{\{1,2\},\{1,3\},\{2,3\},\{4,5\}\}$ a cycle is formed. If $N_9 \in \{\{1,4\},\{1,5\},\{3,4\},\{3,5\}\}$ we obtain a graph isomorphic to (m). If $N_9 \in \{\{2,4\},\{2,5\}\}$ we obtain a graph isomorphic to (r). If $|N_9| = 1$, then resource location $9 \in L$ is obsolete.

Case 2.g. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

By using the (same) reasoning of *Case 1.j* we end up with subgraph (i) and (j).

(i) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4\}$. If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, then $9 \in L$ is an obsolete resource location. For $N_9 = \{5\}$ we obtain a graph isomorphic to (s).

(j) $N_6 = \{1,2\}, N_7 = \{3,4\}, N_8 = \{5\}$. If $|N_9| = 1$, then $9 \in L$ is obsolete.

Case 2.h. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 1.k* we end up with subgraph (k).

(k) $N_6 = \{1,2\}, N_7 = \{3\}, N_8 = \{4\}$. If $N_9 \in \{\{1\},\{2\},\{3\},\{4\}\}$, then resource location $9 \in L$ is obsolete. For $N_9 = \{5\}$ we obtain a graph isomorphic to (t).

Case 2.i. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 1.l* we end up with subgraph (l).

(l) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}$. If $N_9 \in \{\{1\},\{2\},\{3\}\}$, then resource location $9 \in L$ is obsolete. For $N_9 \in \{\{4\},\{5\}\}$ we obtain a graph isomorphic to (u).

Case 3. $|L| = 5$.

Let $L = \{6,7,8,9,10\}$ with $|N_6| \geq |N_7| \geq |N_8| \geq |N_9| \geq |N_{10}|$ without loss of generality.

Case 3.a. $|N_6| \geq 3$.

The reasoning is identical to *cases 2.a, 2.b, 2.c, 2.d, 2.e*.

Case 3.b. $|N_6| = 2, |N_7| = 2, |N_8| = 2$.

By using the (same) reasoning of *Case 2.f* we end up with subgraph $(m),(n),(o),(p),(q)$ and (r).

(m) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{3,5\}$. If $|N_{10}| = 2$ a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(n) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{1,5\}$. If $|N_{10}| = 2$ a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(o) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{1,4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(p) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$, $N_9 = \{2,5\}$. If $|N_{10}| = 2$ a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(q) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{2,4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

(r) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4,5\}$, $N_9 = \{2,4\}$. If $|N_{10}| = 2$ a cycle is formed. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.c. $|N_6| = 2, |N_7| = 2, |N_8| = 1$.

By using the (same) reasoning of *Case 2.g* we end up with subgraph (s).

(s) $N_6 = \{1,2\}, N_7 = \{2,3\}, N_8 = \{4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.d. $|N_6| = 2, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 2.h* we end up with subgraph (t).

(t) $N_6 = \{1,2\}, N_7 = \{3\}, N_8 = \{4\}$, $N_9 = \{5\}$. If $|N_{10}| = 1$, resource location $10 \in L$ is obsolete.

Case 3.e. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 2.i* we end up with subgraph (u).

(u) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}$, $N_9 = \{4\}$. If $N_{10} \in \{1,2,3,4\}$, resource location $10 \in L$ is obsolete. For $N_{10} = \{5\}$ we obtain a graph isomorphic to (v).

Case 4. $|J| \geq 6$.

Let $J = \{6,7,\ldots,k\}$ with $k \geq 12$ and $|N_6| \geq |N_7| \geq \ldots| \geq |N_k|$ without loss of generality.

Case 4.a. $|N_6| \geq 2$.

The reasoning is identical to *cases 3.a, 3.b, 3.c, 3.d*.

Case 4.b. $|N_6| = 1, |N_7| = 1, |N_8| = 1$.

By using the (same) reasoning of *Case 3.e* we end up with subgraph (v).

(v) $N_6 = \{1\}, N_7 = \{2\}, N_8 = \{3\}, N_9 = \{4\}, N_{10} = \{5\}$. Let $|N_{11}| = 1$. Then $11 \in L$ is obsolete.

We conclude that the graph (without cycles and obsolete resource locations) is isomorphic to one of the graphs of Figure 5. $\square$

Proof of Theorem 2

We distinguish in the number of players. For $k \in \{1, 2, 3\}$ the core is non-empty by Theorem 1. Hence, we can restrict attention to $k = 4$ and $k = 5$.

Case a. $k = 4$.

Based on Proposition 2 and Lemma 2, we can focus on MCL situations with two resources only and no obsolete resources in the corresponding graph. Let $\theta \in \Theta$ with $|N| = 4$, $\sum_{i \in N} r_i = 2$ and a corresponding graph $\mathscr{G} = (N, L, E)$ without cycles and obsolete resource locations. Now, we distinguish between two cases.

Case a.1. $|L| \leq 2$.

Take $x_i = p_i$ for all $i \in N$ for which $L_i \neq \emptyset$ and $x_i = 0$ otherwise. Let $S \subseteq N$, then

$$v^\theta(S) \leq \sum_{i \in S: L_i \neq \emptyset} p_i = \sum_{i \in S: L_i \neq \emptyset} x_i = \sum_{i \in S: L_i \neq \emptyset} x_i + \sum_{i \in S: L_i = \emptyset} x_i = \sum_{i \in S} x_i.$$

The inequality is a natural upperbound. The first and third equality hold by definition. The second equality holds as $x_i = 0$ for all $i \in N$ for which $L_i = \emptyset$.

Case a.2. $|L| \geq 3$.

For θ, i.e., the MCL situation under consideration, which has a corresponding graph $\mathscr{G} = (N, L, E)$ isomorphic to one of the graphs of Figure 4 (Lemma 3), one can check easily (see footnote 1 in main text) that (i) matrix A of the corresponding linear programming problem $RMCL^\theta(N)$ in standard LP-form, i.e., in form $Ax \leq b$, is totally unimodular and (ii) vector b has all integer entries. In addition, $RMCL^\theta(N)$ has a finite optimum and so, by Lemma 1, $RMCL^\theta(N)$ has integer optimal solutions. As $0 \leq y_i \leq 1$ for all $i \in N$, it follows directly that $0 \leq x_j \leq 1$ for all $j \in L$ for all optimal solutions of $RMCL^\theta(N)$. Hence, $RMCL^\theta(N)$ has an integer optimal solution with not only $y_i \in \{0,1\}$ for all $i \in N$, but also $x_j \in \{0,1\}$ for all $j \in L$. Based on Proposition 3, the corresponding core of (N, v^θ) is non-empty.

Case b. $k = 5$.

Based on Proposition 2 and Lemma 2, we can focus on MCL situations with two resources and no obsolete resources in the corresponding graph and MCL situations with three resources and no obsolete resources in the corresponding graph. First, we consider the situation with two resources.

Case b.1. $\sum_{i \in N} r_i = 2$.

Let $\theta \in \Theta$ with $|N| = 5$, $\sum_{i \in N} r_i = 2$ and a corresponding graph $\mathscr{G} = (N, L, E)$ without cycles and obsolete resource locations. Now, we distinguish between two cases.

Case b.1.1. $|L| \leq 2$.

Similar to *Case a.1.*

Case b.1.2. $|L| \geq 3$.

Similar to *Case a.2.*, by taking into consideration the graphs of Lemma 4 instead of the graphs of Lemma 3.

Case b.2. $\sum_{i \in N} r_i = 3$.

Let $\theta \in \Theta$ with $|N| = 5$, $\sum_{i \in N} r_i = 3$ and a corresponding graph $\mathscr{G} = (N, L, E)$ without cycles and obsolete resource locations. Now, we distinguish between two cases.

Case b.2.1. $|L| \leq 3$.

Similar to *Case a.1.*

Case b.2.2 $|L| \geq 4$.

Similar to *Case a.2.*, by taking into consideration the graphs of Lemma 4 instead of the graphs of Lemma 3.

We conclude that the cores of MCL games without cycles in the corresponding graph for k players with $k \in \{1, 2, 3, 4, 5\}$ are non-empty. □