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# Collaborative Replenishment in the Presence of Intermediaries

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## Abstract

In complex supply chains, individual downstream buyers would often rather replenish from intermediaries than directly from manufacturers. Direct replenishment from manufacturers can be a less costly alternative when carried out by the buyers collaboratively. This paper constructs a general model to study collaborative replenishment in multi-product supply chains in the presence of intermediaries. We introduce a class of associated cooperative games, outline a sufficient condition for their stability, and formulate a lower bound for individual allocations in the core. Drawing upon a class of associated two-stage games, we investigate the choice of allocation rule and its effect on the individuals' strategic decisions about participation in the collaborative organization. We prove that the Shapley value coordinates the supply chain as it makes complete participation the best individual choice for all buyers under complete or incomplete information. We show if the collaborative organization disregards the replenishment options from the intermediaries, so that they would be handled individually, no allocation rule could always coordinate the supply chain.

## 1 Introduction

*Intermediaries* are economic entities who arbitrate transactions in between upstream suppliers and downstream buyers (Wu, 2004). According to the intermediation theory of the firm (Spulber, 1996), a firm is created when the gains from intermediated exchange exceed the gains from direct exchange. The gains created by intermediaries in many supply chains stem from aggregating demands of competing downstream buyers to achieve economy of scale, and consolidating upstream supply to reduce order and delivery costs. Traditionally, supply chain intermediaries generate these benefits via procuring products, holding inventories, and reselling them at a margin. This paper investigates the possibilities of increasing supply chain efficiency by reducing additional intermediation costs due to double marginalization (Spengler, 1950) and excessive inventory holding costs—an objective that is attainable by *collaboration* among downstream buyers.

The role of supply chain intermediaries are more significant in industries with high degree of product variability, market fragmentation, and sourcing globalization, e.g., in fashion and agro-food sectors as studied in Purvis et al. (2013) and Appel et al. (2014) respectively. This paper is particularly motivated by supply chain intermediation in the automotive after-sale market. The automotive after-market deals with thousands of products, comprises many echelons—e.g., manufacturers, importers, wholesalers, garages, and car owners—and is filled with excessive inventories and inefficiencies at various echelons (AASA, 2012). Collaborative purchasing and replenishment in this context is becoming an emerging trend to reduce costs and improve efficiency (London Economics, 2006).

The enduring presence of intermediaries in certain supply chains implies that individual downstream buyers find it worthwhile to replenish indirectly even though intermediaries charge considerably higher prices than manufacturers. Despite price disparity, replenishing from intermediaries often provides the opportunity to bundle orders for several products and receive them in one delivery, instead of dealing with numerous manufacturers whose minimum volume requirements, fixed ordering costs, or farther geographical distance impose higher replenishment costs and/or longer lead-times. By creating a critical mass, collaborative replenishment and group purchasing could remedy some of the challenges involved in dealing directly with manufacturers.

In this paper, we formally introduce *Collaborative Replenishment in the presence of Intermediaries* (CRI) situations as a general framework for collaborative replenishment of multiple products by several downstream buyers with direct and indirect sourcing options from manufacturers and local intermediaries. The downstream buyers are points of sale to the market. Each manufacturer produces a single product. Intermediaries themselves procure products from the manufacturers, keep them in the stock, and offer the products to the downstream buyers. The natural conditions exerted on cost functions capture the potential conflict between the economies of scale in dealing with direct and indirect replenishment sources: per-product replenishment costs could decrease either if buyers replenish more products from intermediaries or if collaborative replenishments from manufacturers are carried out by more buyers. The collaborative organizations of downstream buyers take advantage of both direct and indirect replenishment sources to achieve the lowest possible costs by selecting the best *replenishment policies* for the participating buyers. Figure 1 depicts an example of a CRI situation with four players who must replenish four products. It also demonstrates a replenishment policy which determines the player-products pairs that would be replenished from either of the sources by the players. For instance, player 2 in this example replenishes products 1 and 2 from the intermediary, product 3 from the manufacturer 3 (jointly with players 3 and 4), and product 4 from the manufacturer 4 (jointly with players 1, 3, and 4). To the best of our knowledge, this paper is the first to consider such mixed policies in collaborative settings.

The starting point in our study of CRI situations is to elaborate on the underlying optimization problems and the optimal replenishment policies. To understand the nature of collaboration among the buyers, we construct a cooperative cost game associated with these situations and study the possibility of achieving stability, i.e., finding allocations in the core (Gillies, 1959) of these games. We extend our analysis by allowing downstream buyers to

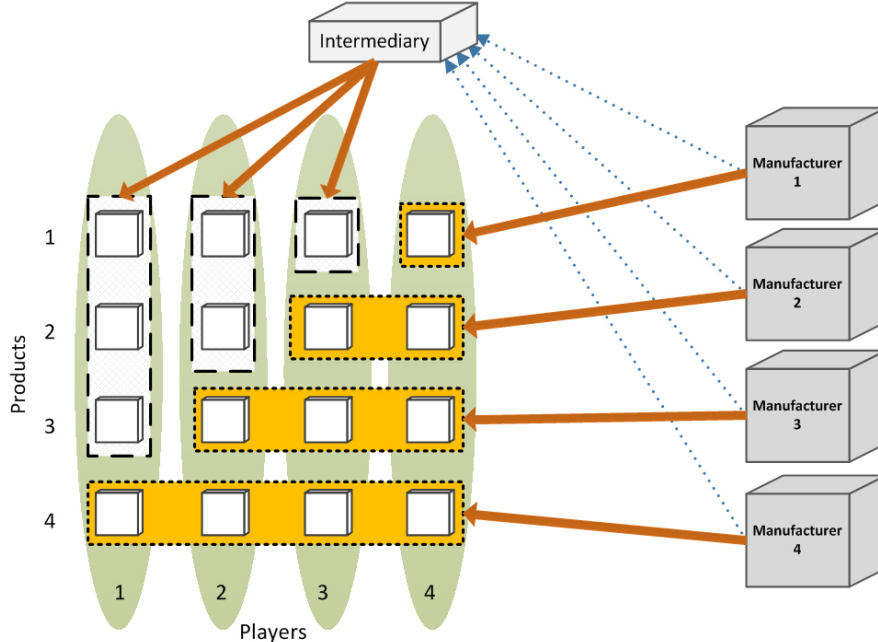


Figure 1: Graphical representation of a replenishment policy

strategically decide about the extend of their participation in the collaborative replenishment organization. In order to do so, we introduce a class of two-stage games associated with CRI situation. In the first stage of a two-stage CRI game the buyers individually choose the products whose replenishment sources will be decided by the collaborative organization. Each buyer replenishes its withheld product set individually from the intermediaries. In the second stage, the cooperative CRI game induced by the participation strategies of the buyers is played where joint costs are divided according a known allocation rule. We examine the choice of allocation rule and its effect on the coordination of the supply chain, i.e., to achieve the minimum total cost of the corresponding centralized system. Finally, we elaborate on the necessity of including indirect replenishment options in the cooperative stage to coordinate the supply chain.

The results obtained in this paper are of two types: (1) results pertaining to general CRI situations, and (2) results pertaining to the class of *submodular CRI situations*, i.e., situations whose total replenishment cost functions for every group of buyers are submodular on their replenishment choice sets. As we prove, the class of submodular CRI situations contains situations wherein the replenishment cost components from intermediaries and manufacturers are themselves submodular. Single-source instances of such joint replenishment models are extensively studied in Meca et al. (2004), Anily and Haviv (2007), Zhang (2009), Van den Heuvel et al. (2007), Hartman et al. (2000), and Özen et al. (2011) among others. Therefore, the second type of results presented in this paper holds for multi-product-multi-source extensions of aforementioned models.

Generally, obtaining the optimal replenishment policies for a group of buyers requires solving a combinatorial optimization problem. For submodular CRI situations, we show

that the optimal replenishment policies exhibit a nested property meaning that if it is optimal for a group of buyers to replenish a product directly from its manufacturer, doing so by those buyers remains optimal in every group of buyers containing the former group. Therefore, direct replenishers of a product never grow smaller as more buyers join the collaborative organization. Although the cooperative CRI games are in general subadditive, the nested property of optimal replenishment policies for submodular CRI situations allows us to prove that their associated cooperative games are concave and their cores are always non-empty. For general CRI games we show that whenever the core is non-empty, every buyer has to pay at least the entire cost of its indirect replenishments from the intermediaries. Thus, core allocations for CRI games never subsidize the indirect replenishment costs of any buyer. In order to divide the joint costs among the buyers, we suggest the Shapley value (Shapley, 1953). While the Shapley value constitutes a core allocation in games associated with submodular CRI situations, it also has the ability to coordinate the supply chain once buyers are allowed to *partially* participate in the collaborative organization.

To formally assess the strategic participation of buyers in the collaborative replenishment organization, we investigate two-stage CRI games. Intuitively, it is to the benefit of the aggregate system that all buyers participate with all their products in the collaborative organization so that the grand coalition could execute the centrally optimal replenishment policies. But individual buyers may choose other strategies if they perceive that partial participation would be to their interest. For general CRI situations, we show that with the Shapley value as the allocation rule for the cooperative stage, individual buyers can never make a better move than adopting the complete participation strategies irrespective of others' strategic moves, that is, the complete participation strategy profile is always a *weakly dominant* strategy profile. In this sense the Shapley value *implements* the centrally optimal replenishment policies in dominant strategies. Maskin and Sjöström (2002) explain that this is the most demanding form of implementation which is often impossible to achieve. As the complete participation strategy of an individual buyer is unaffected by the attributes of the other buyers, it is optimal for each buyer to participate completely in the cooperative organization even if no information about the other buyers is available. We conclude that the Shapley value has the ability to coordinate the supply chain in CRI situations. As the final intuition of the paper, we demonstrate that if indirect replenishment options from intermediaries are disregarded in the collaborative organization, so that the collaborative organization always replenish directly from manufacturers, no allocation rule can guarantee that the supply chain would always be coordinated. Hence, upon availability of indirect replenishment options, collaborative organizations of replenishing buyers must explicitly take those options into consideration if supply chain coordination is sought after.

The rest of this paper is organized as following. In Section 2, we briefly overview the relevant literature. Section 3 contains an overview of main concepts used in this paper. We formally present the CRI situations in Section 4. In Section 5 we discuss the replenishment policies and some of their properties. The cooperative cost games associated with CRI situations are introduced in Section 6 where the corresponding cost-sharing problem is also addressed. The two-stage CRI games are investigated in Section 7. The necessity of including the replenishment options from intermediaries are discussed in Section 8. Section 9 concludes

the paper.

## 2 Literature Review

Several papers in the literature elaborate on the opportunities for consolidating costs, obtaining lower purchase prices, carrying less stocks, and reducing risks of supply/demand uncertainty as the result of collaboration in replenishment and procurement activities. Dror and Hartman (2011) and Fiestras-Janeiro et al. (2011) provide surveys of cooperative and non-cooperative games associated with replenishment and procurement situations.

An important advantage in collaborative replenishment is the possibility of aggregating order and/or delivery costs. Drawing upon basic EOQ model, Meca et al. (2004) introduce the class of inventory games where downstream players aggregate their logistics costs by placing joint orders and show that the total cost is submodular on the set of players. Dror and Hartman (2007) extend the basic inventory game to the setting which takes into account the player-specific order costs in the joint replenishment process. They show that collaborative replenishment may not necessarily be beneficial if players could only place joint orders simultaneously. However, Anily and Haviv (2007) prove that if replenishment policies follow the *powers-of-two* (Jackson et al., 1985) structure, so that downstream players are not forced to synchronize all of their orders, the collaborative replenishment is always beneficial and the total cost is submodular on the set of players. Zhang (2009) extends this result to situations where players are allowed to have a joint inventory stocking point and obtains similar results. Van den Heuvel et al. (2007) introduce and investigate the class of economic lot-sizing games wherein players face periodic, yet deterministic, demand and have the option to place joint orders. They introduce cases in which the joint cost function is submodular. Timmer et al. (2013) extend the model in Meca et al. (2004) to Poisson demand and conjecture the submodularity of the corresponding cost function.

The collaborative replenishment problem has also been investigated in settings with strategic players. Meca et al. (2003) study a single-item inventory game in strategic form with players announcing their desired replenishment cycles to an intermediary who places orders with the manufacturer. Alternative games with players announcing their contribution to ordering costs are investigated by Körpeoğlu et al. (2012) and Körpeoğlu et al. (2013). The latter models allow players to be privately informed about their types. Finally, Bylka (2011) analyze an inventory batching game in strategic form and describe the structure of Nash equilibria.

In addition to consolidating fixed costs, collaborative replenishment can also reduce the risks associated with stochastic demands. The extensive line of research on risk pooling in inventory management and procurement starts with the work of Hartman et al. (2000) and in the context of newsvendor problem. Slikker et al. (2005) further study these situations while allowing downstream players to transship unused products amongst themselves and show that allocations in the core always exists. Özen et al. (2011) particularly study situations where the corresponding collaborative replenishment models have submodular cost functions. Montrucchio et al. (2012) provide a review of cooperative newsvendor games. Infinite-horizon versions of inventory risk pooling games are studied in Karsten et al. (2012) and Karsten

and Basten (2014) in the context of expensive and low-demand spare parts.

Another stream of research focuses on the cost-sharing problems in collaborative purchasing organizations that take advantage of suppliers' discount schedules. Nagarajan et al. (2009) compare some of the well-known allocations for dividing the joint costs in such situations. Schotanus et al. (2008) discuss the unfairness of the equal price allocation method in purchasing groups. Schaarsberg et al. (2013) introduce and analyze the class of maximum collaborative purchasing situations and their associated games where the purchase price of a group of players is determined by the largest order quantity of the players in the group. In the context of health-care supply chains, the effect of group purchasing organizations on distribution of profit and providers' total purchasing cost have been investigated in Hu et al. (2012).

A number of papers in the operations management literature investigate two-stage games with a non-cooperative first stage game played in anticipation of a related cooperative game played in the second stage. Brandenburger and Stuart (2007) provide an axiomatic approach to these games which they refer to as biform games. Stuart (2005) use the biform game structure to investigate the pricing decisions following the inventory decisions among a group of competing newsvendors. In the context of inventory pooling and transshipments, Anupindi et al. (2001) study the choice of allocation rules for the cooperative game in second stage and its effect on the first stage strategies. They show that the use of dual allocations (Owen, 1975) makes the centrally optimal order quantities a Nash equilibrium (Nash, 1950) in the first stage non-cooperative game. Including the supplier into the analysis, Kemahlioglu-Ziya and Bartholdi (2011) show that with the Shapley value as the allocation rule, the retailers have incentive to join the inventory pooling coalition and the supplier carries the level of inventory that is optimal for the coalition. Özen et al. (2008) study a two-stage inventory pooling game with warehouses and show that the set of payoff vectors resulting from strong Nash equilibria corresponds to the core of the cooperative game played in the second stage.

### 3 Preliminaries

**Set functions** Given a finite set  $\Omega$ , and its power set  $\wp(\Omega)$ ,  $f : \wp(\Omega) \rightarrow \mathbb{R}$  is a *set function* that gives real values to subsets of  $\Omega$ . The following properties of set functions are of interest:

- $f$  is *non-decreasing* if for every  $A \subset B \subseteq \Omega$  we have  $f(A) \leq f(B)$ .
- $f$  is *subadditive* if for every  $A, B \subset \Omega$ ,  $A \cap B = \emptyset$ , we have  $f(A \cup B) \leq f(A) + f(B)$ .
- $f$  is *submodular* if for every  $A \subseteq B \subset \Omega$  and every element  $a \in \Omega \setminus B$  it holds that  $f(B \cup a) - f(B) \leq f(A \cup a) - f(A)$ .<sup>1</sup>

The returned value of a non-decreasing set function never decreases as the result of including more elements. Subadditivity limits the amount of increase due to including more elements so that the value of union of two disjoint sets does not exceed their sum. A submodular set

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<sup>1</sup>For notational convenience we do not use braces for union and exclusion of single element sets. That is, we write  $A \cup a$  instead of  $A \cup \{a\}$  and  $A \setminus a$  instead of  $A \setminus \{a\}$ .

function demonstrates a diminishing returns property which makes it analogous to concave continuous functions.

**Cooperative games** A Transferable Utility (TU) cooperative cost game is a pair  $(N, c)$  where  $N$  is a finite set of players and  $c : \wp(N) \rightarrow \mathbb{R}$  a set function with  $c(\emptyset) = 0$  that determines the cost to be paid by each group of players. The game  $(N, c)$  is *subadditive* if  $c$  is subadditive on the set of players and it is *concave* if  $c$  is submodular on the set of players. An *allocation* for players in  $N$  is  $\beta = (\beta_i)_{i \in N}$  such that  $\beta_i \in \mathbb{R}$  for every  $i \in N$ . An allocation  $\beta$  is *efficient* for  $(N, c)$  if  $\sum_{i \in N} \beta_i = c(N)$ . An allocation rule is *stable* for  $(N, c)$  if for any  $S \subseteq N$  it holds that  $\sum_{i \in S} \beta_i \leq c(S)$ . The *core* of a game contains all of its efficient and stable allocations. An allocation in the core provides sufficient incentives for all players not to break apart from the grand coalition while dividing the total cost entirely among players.

**Non-cooperative games** A cost game in strategic form is a triple  $(N, A, z)$  where  $N$  denotes the set of players,  $A = (A_i)_{i \in N}$  is the vector of strategy sets of players and  $z = (z_i)_{i \in N}$  is the vector of player-specific cost functions which assign values to every *strategy profile*  $L = (L_i)_{i \in N}$  with  $L_i \in A_i$  for every  $i \in N$ . For  $S \subseteq N$ , let  $L_S$  be the reduction of  $L$  to players in  $S$  and let  $L_{-S}$  be the reduction of  $L$  to players in  $N \setminus S$ . The following strategy profiles are of interest in this paper:

- $L$  is a *Nash equilibrium* if for every  $i \in N$  and every  $L'_i \in A_i$  it holds that  $z_i(L) \leq z_i(L'_i, L_{-i})$ .
- $L$  is a *weakly dominant* strategy profile if for every  $i \in N$  and every  $L' \in \prod_{i \in N} A_i$  it holds that  $z_i(L_i, L'_{-i}) \leq z_i(L')$ .

Unilateral deviations from a Nash equilibrium does not reduce the cost of any players. A weakly dominant strategy for a player is its best choice of strategy irrespective of other players' choices. The last concept is a refinement of Nash equilibrium meaning that if  $L$  is a weakly dominant strategy profile, it is also a Nash equilibrium. The reverse does not hold necessarily.

## 4 Mathematical Model

Consider a supply chain with a set of downstream buyers, hereafter the *players*, represented by the index set  $N = \{1, \dots, n\}$ , replenishing a variety of different products to sell in their local markets. The set of products replenished by a player  $i \in N$  is denoted by  $E_i$ . The vector  $E = (E_i)_{i \in N}$  denotes the player-specific product sets. Each product is produced and sold by a distinct manufacturer. Thus the set of all products, i.e.,  $\mathcal{E} = \bigcup_{i \in N} E_i$ , also represents the set of manufacturers. In addition to the manufacturers, supply chain intermediaries, e.g., regional wholesalers or volume distributors, also sell some or all products in  $\mathcal{E}$ . The intermediaries by themselves procure products from the manufacturers and keep them in stock. The players have the option to obtain each product either from its corresponding manufacturer or from an intermediary who sells it.



## 4.1 Replenishments from the Intermediaries

When a player  $i \in N$  replenishes a subset of its products  $L_i \subseteq E_i$  from the intermediaries, it will do so by optimally choosing corresponding decision variables—i.e., batch sizes, ordering cycles, selection of intermediaries etc.—to attain the minimum possible per-period replenishment cost. We refrain from the operational details at this level and instead introduce the *indirect replenishment cost function*  $r_i^w : \wp(E_i) \rightarrow \mathbb{R}$  that gives the minimum per-period replenishment cost from intermediaries of player  $i$  for subsets of products. We let  $r_i^w(\emptyset) = 0$  for every  $i \in N$ . The vector  $r^w = (r_i^w)_{i \in N}$  denotes indirect replenishment cost functions for all players.

It is natural to assume that for every  $i \in N$  the indirect replenishment cost function  $r_i^w$  is *non-decreasing* and *subadditive* on its product set  $E_i$ . The first condition reflects the fact that replenishing more products never results in a reduction in costs. The second condition asserts that by combining the replenishments of multiple sets of products from the intermediaries, their total per-period replenishment cost does not increase—although in practice it is usually the case that joint replenishments of multiple products provide opportunities to obtain additional savings by taking advantage of discounts or batch deliveries. The non-decreasing condition of indirect cost functions is not formally needed in this paper.

We assume that replenishment costs from intermediaries are additive over the set of players, that is, intermediaries cater to players on the individual bases and no savings can be obtained by combining the indirect replenishments of different players. This justifies our expression of indirect replenishment cost functions in terms of individual players.

## 4.2 Replenishments from the Manufacturers

Every product can be replenished directly from its manufacturer. Although factors such as high order costs may render direct replenishments unattractive for individual players, by aligning replenishment cycles and placing joint orders, groups of players could obtain savings when replenishing a product directly from its manufacturer. When a group of players jointly replenish a product from its manufacturer, corresponding decision variables would be chosen to minimize the per-period replenishment cost of that product for the group. Abstracting away from the operational details, we introduce the *direct replenishment cost function*  $r_l^m : \wp(N) \rightarrow \mathbb{R}$  as the set function that obtains the minimum per-period replenishment cost from manufacturer of product  $l \in \mathcal{E}$  for different groups of players. We let  $r_l^m(\emptyset) = 0$  for every  $l \in \mathcal{E}$ . The vector  $r^m = (r_l^m)_{l \in \mathcal{E}}$  denotes direct replenishment cost functions for all products.

Similar to the previous case, we impose basic conditions on direct replenishment cost functions. For every  $l \in \mathcal{E}$ , we require  $r_l^m$  to be *non-decreasing* and *subadditive* on the set of players  $N$ . That is, including more players in joint orders from a manufacturer never results in lower total costs although it can be cheaper to replenish a product directly as a single group instead of replenishing it in separate groups.

As each manufacturer produces a distinct product, we assume that direct replenishment costs from the manufacturers are additive over the set of products. Thus orders for multiple products from different manufacturers cannot be consolidated to make any savings. This explains our expression of direct replenishment cost functions in terms of distinct products.

### 4.3 CRI Situations

We introduce CRI situations in order to succinctly encapsulate the relevant information necessary to formalize the settings described above. An instance of CRI situations can be described by the tuple

$$\Gamma = (N, E, r^w, r^m)$$

with its elements defined as above. The vector of indirect and direct replenishment costs,  $r^w$  and  $r^m$ , are referred to as the *cost components* of the situation. The set of all CRI situations with player set  $N$  is denoted by  $\mathbf{\Gamma}$ .

## 5 Replenishment Policies

Replenishment policies, which represent the various choices regarding the replenishment sources of different products for different players, are the main decision variables in CRI situations. In this paper we assume that the choices of replenishment sources of all products and all players are binary, i.e., every single product required by every player is sourced entirely either from an intermediary or its corresponding manufacturer. Thus, in order to completely describe the replenishment actions of all players with regard to all products, it is sufficient to underline the replenishments from one of the sources only.

Let  $\Gamma = (N, E, r^w, r^m)$  be an arbitrary CRI situation. We define the *replenishment choice set* of a player  $i$ ,  $i \in N$ , as the set of all player-product pairs specific to  $i$  and denote it by

$$\mathcal{X}_i^\Gamma = \{(i, l) | l \in E_i\}.$$

The replenishment choice sets for groups of players are obtained accordingly by concatenating their individual choice sets. For every  $S \subseteq N$ , we denote the replenishment choice set of  $S$  by

$$\mathcal{X}_S^\Gamma = \bigcup_{i \in S} \mathcal{X}_i^\Gamma.$$

We define a *replenishment policy*,  $X$ , as a collection of player-product pairs that are replenished directly *from the manufacturers*. A replenishment policy  $X$  is *feasible* for players in  $S \subseteq N$  whenever  $X \subseteq \mathcal{X}_S^\Gamma$ . Note that with this definition a feasible replenishment policy for a subset of players is also feasible for other subsets of players which contain the former players. However, the reverse does not hold necessarily.

In order to develop the total replenishment cost in CRI situations we define two auxiliary functions that explicitly determine the replenishment actions of groups of players. Given a replenishment policy  $X$  and a product  $l \in \mathcal{E}$ , the *direct replenishers of  $l$  in  $X$* , i.e., individual players who obtain the product  $l$  from its corresponding manufacturer, are denoted by

$$I_l^\Gamma[X] = \{i \in N | (i, l) \in X\}.$$

A replenishment policy readily reveals the products that are replenished from the manufacturers by the players. The other element needed for calculating the total cost of a

replenishment policy is the set of products that each player replenishes from the intermediaries. This can be obtained by excluding the directly replenished products of a player from its specific product set. Given a replenishment policy  $X$  and a player  $i \in N$ , the *indirectly replenished products of  $i$  in  $X$*  are denoted by

$$P_i^\Gamma[X] = \{l \in E_i \mid (i, l) \notin X\}.$$

We are now ready to calculate the total replenishment cost associated with a replenishment policy for a subset of players. For every  $S \subseteq N$ , we define the *replenishment cost function for  $S$* ,  $r_S^\Gamma : \wp(\mathcal{X}_S^\Gamma) \rightarrow \mathbb{R}$ , such that for every feasible replenishment policy for  $S$ ,  $X \subseteq \mathcal{X}_S^\Gamma$ , we have

$$r_S^\Gamma(X) = \sum_{i \in S} r_i^w(P_i^\Gamma[X]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X]). \quad (1)$$

Hence the cost of a given replenishment policy  $X$  for  $S$  is the sum of replenishment costs from intermediaries of players in  $S$  for their indirectly replenished products in  $X$  plus the sum of replenishment costs from manufacturers of all products for their direct replenishers in  $X$ . The following lemma illustrates a relation between the costs of a feasible replenishment policy for two subsets of players.

**Lemma 1.** *Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$  and consider  $S \subset T \subseteq N$ . Let  $X$  be a feasible replenishment policy for  $S$ . We have  $r_T^\Gamma(X) = \sum_{i \in T \setminus S} r_i^w(E_i) + r_S^\Gamma(X)$ .*

*Proof.* Since  $X \subseteq \mathcal{X}_S^\Gamma$ , for every player  $i \in T \setminus S$  there exists no  $l \in \mathcal{E}$  such that  $(i, l) \in X$ . Therefore, for every  $i \in T \setminus S$  we have  $P_i^\Gamma[X] = E_i$ . Consequently, from definition of  $r^\Gamma$  in (1) we have

$$\begin{aligned} r_T^\Gamma(X) &= \sum_{i \in T} r_i^w(P_i^\Gamma[X]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X]) \\ &= \sum_{i \in T \setminus S} r_i^w(E_i) + \sum_{i \in S} r_i^w(P_i^\Gamma[X]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X]) \\ &= \sum_{i \in T \setminus S} r_i^w(E_i) + r_S^\Gamma(X). \end{aligned}$$

□

Lemma 1 allows one to evaluate the cost of a replenishment policy  $X$  that is feasible for  $S \subset N$  for its supersets. To do so, indirect replenishment costs of the entire product sets of extra players must be added to the replenishment cost of  $X$  for  $S$ .

An *optimal replenishment policy* for a subset of players has the lowest replenishment cost among all feasible replenishment policies for those players. The cost of an optimal replenishment policy for  $S \subseteq N$  is denoted by:

$$c^\Gamma(S) = \min_{X \subseteq \mathcal{X}_S^\Gamma} r_S^\Gamma(X) \quad (2)$$

## 5.1 Submodular CRI Situations

We call a CRI situation submodular if the replenishment cost function of every group of players is submodular on its replenishment choice set. The following definition formalizes this.

**Definition 1.** A CRI situation  $\Gamma \in \mathbf{\Gamma}$  is submodular if for every  $S \subseteq N$ ,  $r_S^\Gamma$  is submodular on  $\mathcal{X}_S^\Gamma$ .

We let  $\mathbf{\Gamma}^{sm} \subset \mathbf{\Gamma}$  be the set of all submodular CRI situations. Submodularity of a CRI situation has interesting consequences which we will discuss below in this paper. Before elaborating on such consequences, however, we present a critical observation with regard to a sufficient condition for submodularity of a CRI situation—a condition which enable us to extend many single-source joint replenishment models in the literature.

**Theorem 1.** Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$  with  $r_i^w$  submodular on  $E_i$  for every  $i \in N$  and  $r_l^m$  submodular on  $N$  for every  $l \in \mathcal{E}$ . We have  $\Gamma \in \mathbf{\Gamma}^{sm}$ .

*Proof.* Fix  $S \subseteq N$ . Let  $X, X' \subseteq \mathcal{X}_S^\Gamma$  be arbitrary feasible replenishment policies of  $S$  such that  $X' \subseteq X$ . Consider a player-product pair  $(j, h)$  with  $j \in S$ ,  $h \in E_j$ , and  $(j, h) \in \mathcal{X}_S^\Gamma \setminus X$ . The replenishment cost function  $r_S^\Gamma$  is submodular on  $\mathcal{X}_S^\Gamma$  if

$$r_S^\Gamma(X \cup (j, h)) - r_S^\Gamma(X) \leq r_S^\Gamma(X' \cup (j, h)) - r_S^\Gamma(X'). \quad (3)$$

We continue in two steps:

(Step 1) By definition of  $P_i^\Gamma$  we have  $P_i^\Gamma[X \cup (j, h)] = P_i^\Gamma[X]$  for every  $i \in S \setminus j$ , and  $P_j^\Gamma[X \cup (j, h)] = P_j^\Gamma[X] \setminus h$ . Similar statements hold for  $X'$  as well. The assumption  $X' \subseteq X$  implies that for every  $i \in S$  we have  $P_i^\Gamma[X'] \supseteq P_i^\Gamma[X]$ . Submodularity of  $r_j^w$  on  $E_j$  implies that

$$r_j^w(P_j^\Gamma[X']) - r_j^w(P_j^\Gamma[X'] \setminus h) \leq r_j^w(P_j^\Gamma[X]) - r_j^w(P_j^\Gamma[X] \setminus h)$$

or equivalently

$$r_j^w(P_j^\Gamma[X] \setminus h) - r_j^w(P_j^\Gamma[X]) \leq r_j^w(P_j^\Gamma[X'] \setminus h) - r_j^w(P_j^\Gamma[X']). \quad (4)$$

Adding  $\sum_{i \in S \setminus j} r_i^w(P_i^\Gamma[X \cup (j, h)] - P_i^\Gamma[X]) = 0$  and  $\sum_{i \in S \setminus j} r_i^w(P_i^\Gamma[X' \cup (j, h)] - P_i^\Gamma[X']) = 0$  to the left and right sides of (4) respectively obtains

$$\sum_{i \in S} r_i^w(P_i^\Gamma[X \cup (j, h)]) - r_i^w(P_i^\Gamma[X]) \leq \sum_{i \in S} r_i^w(P_i^\Gamma[X' \cup (j, h)]) - r_i^w(P_i^\Gamma[X']). \quad (5)$$

(Step 2) By definition of  $I_l^\Gamma$  we have  $I_l^\Gamma[X \cup (j, h)] = I_l^\Gamma[X]$  for every  $l \in \mathcal{E} \setminus h$ , and  $I_h^\Gamma[X \cup (j, h)] = I_h^\Gamma[X] \cup j$ . Similar statements hold for  $X'$  as well. The assumption  $X' \subseteq X$  implies that for every  $l \in \mathcal{E}$  we have  $I_h^\Gamma[X'] \subseteq I_h^\Gamma[X]$ . On the other hand submodularity of  $r_h^m$  on  $N$  yields

$$r_h^m(I_h^\Gamma[X] \cup j) - r_h^m(I_h^\Gamma[X]) \leq r_h^m(I_h^\Gamma[X'] \cup j) - r_h^m(I_h^\Gamma[X']).$$

By adding  $\sum_{l \in \mathcal{E} \setminus h} r_l^m(I_l^\Gamma[X \cup (j, h)]) - r_l^m(I_l^\Gamma[X]) = 0$  and  $\sum_{l \in \mathcal{E} \setminus h} r_l^m(I_l^\Gamma[X' \cup (j, h)]) - r_l^m(I_l^\Gamma[X']) = 0$  to the left and right sides of the above inequality respectively we get

$$\sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X \cup (j, h)]) - r_l^m(I_l^\Gamma[X]) \leq \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X' \cup (j, h)]) - r_l^m(I_l^\Gamma[X']). \quad (6)$$

To conclude the proof, add (5) and (6) to get

$$\begin{aligned} & \sum_{i \in S} r_i^w(P_i^\Gamma[X \cup (i, h)]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X \cup (i, h)]) - \left( \sum_{i \in S} r_i^w(P_i^\Gamma[X]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X]) \right) \\ & \leq \sum_{i \in S} r_i^w(P_i^\Gamma[X' \cup (i, h)]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X' \cup (i, h)]) - \left( \sum_{i \in S} r_i^w(P_i^\Gamma[X']) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X']) \right). \end{aligned}$$

which is equivalent to (3). Thus  $r_S^\Gamma$  is submodular on  $\mathcal{X}_S^\Gamma$ .  $\square$

According to Theorem 1, submodularity of the cost components is a sufficient condition for a CRI situation to be submodular. The last observation is a significant result as it implies that CRI situations whose components follow several single-source joint replenishment models in the literature are submodular. These models include, but are not limited to, deterministic joint replenishment problems discussed in Meca et al. (2004), Anily and Haviv (2007), Zhang (2009), special cases in Van den Heuvel et al. (2007), as well as stochastic models considered in Hartman et al. (2000) and Özen et al. (2011). As we elaborated in the literature review section, the cost functions in the latter models are submodular.

The submodularity of CRI situations provides some immediate insights with regard to the benefits of joint replenishments from the manufacturers. Assume that for two replenishment policies  $X$  and  $X'$  and an arbitrary player  $i \in N$  it holds that  $P_i^\Gamma[X] = P_i^\Gamma[X']$  and  $I_l^\Gamma[X] \supseteq I_l^\Gamma[X']$  for every  $l \in \mathcal{E}$ . This means that while player  $i$  does exactly the same in  $X'$  as in  $X$ , direct replenishers of any product  $l$  in  $X$  include direct replenishers of  $l$  in  $X'$  as well. Then it follows from the submodularity of  $r_N^\Gamma$  that even if including  $i$  to  $I_h^\Gamma[X']$ ,  $h \in E_i$ , does not obtain a less costly replenishment policy than  $X'$ , including  $i$  to  $I_h^\Gamma[X]$  could obtain a less costly replenishment policy than  $X$ . When  $I_h^\Gamma[X] \supset I_h^\Gamma[X']$  this means that, all other things held constant, the larger the set of direct replenishers of a product, the more likely that the inclusion of a new player yields a less costly policy. This demonstrates the economy of scale in direct replenishments from the manufacturers. Also, inclusion of  $i$  to  $I_h^\Gamma[X]$  could obtain a less costly replenishment policy than  $X$  (even if this is not the case in  $X'$ ) when  $I_h^\Gamma[X] = I_h^\Gamma[X']$  and for some  $l \in \mathcal{E} \setminus h$  we have  $I_l^\Gamma[X] \supset I_l^\Gamma[X']$ . Thus, expansion of the direct replenishers of any product may render direct replenishments profitable in general. This reflects a spill-over effect with regard to the products replenished from the manufacturers.

The submodularity of CRI situations also has important consequences with regard to the tractability of the optimization problem in (2). Grötschel et al. (1988) show that for a submodular function, the Ellipsoid method can be used to construct a strongly polynomial algorithm for its minimization. Hence, submodularity of a CRI situations implies that the optimal replenishment policies can be found efficiently in CRI situations. In the remainder of this section we elaborate on certain properties of optimal replenishment policies in

submodular CRI situations.

**Lemma 2.** *Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}^{sm}$ . The following two statements hold:*

- (i) *If alternative optimal replenishment policies exist for  $S \subseteq N$ , their union is also an optimal replenishment policy for  $S$ .*
- (ii) *Let  $X_S^*$  be an optimal replenishment policy for  $S \subseteq N$ . For every  $T \subseteq N$ ,  $T \supset S$ , there exists an optimal replenishment policy  $X_T^*$  such that  $X_T^* \supseteq X_S^*$ .*

*Proof.* (i) Let  $X^*$  and  $\hat{X}^*$  be alternative optimal replenishment policies for  $S \subseteq N$ . By the submodularity assumption it holds that

$$\begin{aligned}
r_S^\Gamma(X^* \cup \hat{X}^*) - r_S^\Gamma(X^*) &= r_S^\Gamma(X^* \cup (\hat{X}^* \setminus X^*)) - r_S^\Gamma(X^*) \\
&\leq r_S^\Gamma((\hat{X}^* \cap X^*) \cup (\hat{X}^* \setminus X^*)) - r_S^\Gamma(\hat{X}^* \cap X^*) \\
&= r_S^\Gamma(\hat{X}^*) - r_S^\Gamma(\hat{X}^* \cap X^*) \\
&\leq 0
\end{aligned}$$

where the last inequality follows by the assumption that  $\hat{X}^*$  is an optimal replenishment policy for  $S$ . Therefore  $r_S^\Gamma(X^* \cup \hat{X}^*) \leq r_S^\Gamma(X^*)$  which implies that  $X^* \cup \hat{X}^*$  is also an optimal replenishment policy.

(ii) Fix  $T$  as above and assume that for an arbitrary optimal replenishment policy for  $T$ , i.e.,  $X_T^*$ , we have  $X_S^* \setminus X_T^* \neq \emptyset$ . Let  $X_T = X_T^* \cup X_S^*$ . Clearly  $X_T$  is a feasible policy for  $T$  and furthermore  $X_T \supseteq X_S^*$ . We show that  $X_T$  has the lowest possible cost among all other feasible policies. We have

$$\begin{aligned}
r_T^\Gamma(X_T) - r_T^\Gamma(X_T^*) &= r_T^\Gamma(X_T^* \cup [X_S^* \setminus X_T^*]) - r_T^\Gamma(X_T^*) \\
&\leq r_T^\Gamma([X_S^* \cap X_T^*] \cup [X_S^* \setminus X_T^*]) - r_T^\Gamma(X_S^* \cap X_T^*) \\
&= r_T^\Gamma(X_S^*) - r_T^\Gamma(X_S^* \cap X_T^*) \\
&= r_S^\Gamma(X_S^*) - r_S^\Gamma(X_S^* \cap X_T^*) \\
&\leq 0.
\end{aligned}$$

First equality uses the fact that  $X_T = X_T^* \cup [X_S^* \setminus X_T^*]$ . Subsequent inequality follows from submodularity of  $r_T^\Gamma$ . Second equality holds as  $X_S^* = [X_S^* \cap X_T^*] \cup [X_S^* \setminus X_T^*]$ . Since  $X_S^*$  and  $X_S^* \cap X_T^*$  both are feasible policies for  $S$ , last equality can be obtained by using Lemma 1. Final inequality follows from the optimality of  $X_S^*$  for  $S$ . Therefore  $r_T^\Gamma(X_T) - r_T^\Gamma(X_T^*) \leq 0$ . By assumption,  $X_T^*$  is an optimal replenishment policy for  $T$ , thus it can only be the case that  $r_T^\Gamma(X_T) = r_T^\Gamma(X_T^*)$  which implies that  $X_T$  is also an optimal replenishment strategy for  $T$ .  $\square$

The first part of Lemma 2 states that the union of two optimal replenishment policies for a group of players is in itself another optimal replenishment policy. Thus, it can be inferred that for every group of players, there exists an optimal replenishment policy with the most number of player-product pairs replenished from the manufacturers. The second part of

Lemma 2 shows a *nested property* in growing subsets of players. That is, if it is optimal for a subset of players to collectively replenish certain products from their manufacturers, it would also be optimal that this subset of players keep on doing the same in any other subset that contains the former players. The latter can be interpreted in an alternative way: in submodular CRI situations, the set of direct replenishers of a product never shrink as the result of including more players to the collaborative organization. A direct consequence of the nested property of optimal replenishment policies in Lemma 2 is that in submodular CRI situations the optimal policies for larger subsets of players can be built upon those of the smaller subsets.

## 6 Cooperative CRI Games

In this section we study the collaboration among players in CRI situations with the help of a class of cooperative cost games associated with these situations. The cooperative cost games associated with CRI situations, hereafter *cooperative CRI games*, can be constructed by considering the set of players  $N$  and defining the characteristics function to be the optimal replenishment cost function. Thus, for every CRI situation  $\Gamma \in \mathbf{\Gamma}$ , one can define an associated cooperative cost game by  $(N, c^\Gamma)$  where for every  $S \subseteq N$ ,  $c^\Gamma(S)$  is defined as in equation (2). The next theorem exhibits the subadditivity of general CRI games.

**Theorem 2.** *For every  $\Gamma \in \mathbf{\Gamma}$ , the associated cooperative game  $(N, c^\Gamma)$  is subadditive.*

*Proof.* Let  $\Gamma$  be a CRI situation and  $(N, c^\Gamma)$  its associated cooperative game. Consider  $S, T \subseteq N$  such that  $S \cap T = \emptyset$  and let  $X_S^*$  and  $X_T^*$  be optimal replenishment policies for  $S$  and  $T$  respectively. Let  $X = X_S^* \cup X_T^*$  and observe that  $X$  is a feasible replenishment policy for  $S \cup T$ . By definition of  $P_i^\Gamma$  it follows that for every  $i \in S$  we have  $P_i^\Gamma[X] = P_i^\Gamma[X_S^*]$  and for every  $i \in T$  we have  $P_i^\Gamma[X] = P_i^\Gamma[X_T^*]$ . By definition of  $I_l^\Gamma$ , on the other hand, it follows that for every  $l \in \mathcal{E}$  we have  $I_l^\Gamma[X] = I_l^\Gamma[X_S^*] \cup I_l^\Gamma[X_T^*]$ . Thus we have

$$\begin{aligned}
c^\Gamma(S \cup T) &\leq r_{S \cup T}^\Gamma(X) \\
&= \sum_{i \in S} r_i^w(P_i^\Gamma[X_S^*]) + \sum_{i \in T} r_i^w(P_i^\Gamma[X_T^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X_S^*] \cup I_l^\Gamma[X_T^*]) \\
&\leq \sum_{i \in S} r_i^w(P_i^\Gamma[X_S^*]) + \sum_{i \in T} r_i^w(P_i^\Gamma[X_T^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X_S^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X_T^*]) \\
&= r_S^\Gamma(X_S^*) + r_T^\Gamma(X_T^*) = c^\Gamma(S) + c^\Gamma(T)
\end{aligned}$$

where the last inequality follows from the subadditivity of  $r_l^m$ .  $\square$

Subadditivity of CRI games implies that the optimal replenishment cost for the case where all players are participating in the collaborative organization is never higher than the sum of the costs of any other partitionings of the players into independent collaborative organizations. Notice that the proof of Theorem 2 only uses the subadditivity of direct replenishment cost functions and does not require the subadditivity of indirect replenishment costs.

## 6.1 Concavity of CRI Games

In a concave game the contributions of players to the cost of growing subsets of players are non-increasing. The concavity of a cooperative game has important implications with regard to its stability which will be discussed in the next section. Our main result in this section asserts that cooperative games associated with submodular CRI situations are concave. Remember that for a submodular CRI situation the replenishment cost function of each coalition is submodular on its corresponding replenishment choice set while concavity of the associated game requires that the optimal replenishment cost function be submodular on the set of players. Thus, proving that the former implies that latter is non-trivial.

**Theorem 3.** *For every  $\Gamma \in \mathbf{\Gamma}^{sm}$ , the associated cooperative game  $(N, c^\Gamma)$  is concave.*

*Proof.* Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}^{sm}$ . Consider a player  $i \in N$  and let  $S \subset T \subseteq N \setminus i$ . In the rest of the proof we use the shorthand notation  $S^i = S \cup i$ . Let  $X_{T^i}^*$ ,  $X_T^*$ ,  $X_{S^i}^*$ , and  $X_S^*$  be the optimal replenishment policies for  $T^i$ ,  $T$ ,  $S^i$ , and  $S$  respectively in such a way that  $X_{T^i}^* \supseteq X_T^* \supseteq X_S^*$ , and  $X_{T^i}^* \supseteq X_{S^i}^* \supseteq X_S^*$ . From Lemma 2, parts (i) and (ii), we know that such optimal replenishment policies always can be found. Since  $X_{T^i}^*$  is an optimal replenishment policy for  $T^i$ , it is at most as costly as any other feasible replenishment policy for  $T^i$ . By the fact that  $X_T^* \cup [X_{S^i}^* \setminus X_T^*] \subseteq \mathcal{X}_{S^i}^\Gamma$ , it can be inferred that  $X_T^* \cup [X_{S^i}^* \setminus X_T^*]$  is a feasible replenishment policy for  $T^i$ . Therefore:

$$\begin{aligned}
r_{T^i}^\Gamma(X_{T^i}^*) - r_{T^i}^\Gamma(X_T^*) &\leq r_{T^i}^\Gamma(X_T^* \cup [X_{S^i}^* \setminus X_T^*]) - r_{T^i}^\Gamma(X_T^*) \\
&\leq r_{T^i}^\Gamma([X_{S^i}^* \cap X_T^*] \cup [X_{S^i}^* \setminus X_T^*]) - r_{T^i}^\Gamma(X_{S^i}^* \cap X_T^*) \\
&= r_{T^i}^\Gamma(X_{S^i}^*) - r_{T^i}^\Gamma(X_{S^i}^* \cap X_T^*) \\
&= r_{S^i}^\Gamma(X_{S^i}^*) - r_{S^i}^\Gamma(X_{S^i}^* \cap X_T^*) \\
&\leq r_{S^i}^\Gamma(X_{S^i}^*) - r_{S^i}^\Gamma(X_S^*)
\end{aligned}$$

Second inequality follows from the fact that  $X_{S^i}^* \cap X_T^* \subseteq X_{T^i}^*$  and drawing upon the submodularity of  $r_{T^i}^\Gamma$  (by Theorem 1). First equality follows since  $[X_{S^i}^* \cap X_T^*] \cup [X_{S^i}^* \setminus X_T^*] = X_{S^i}^*$ . Second equality holds as both  $X_{S^i}^*$  and  $X_{S^i}^* \cap X_T^*$  are feasible policies for  $S^i$  and by Lemma 1 we have  $r_{T^i}^\Gamma(X_{S^i}^*) = r_{S^i}^\Gamma(X_{S^i}^*) + \sum_{j \in T^i \setminus S^i} r_j^w(E_j)$  and  $r_{T^i}^\Gamma(X_{S^i}^* \cap X_T^*) = r_{S^i}^\Gamma(X_{S^i}^* \cap X_T^*) + \sum_{j \in T^i \setminus S^i} r_j^w(E_j)$ . Last inequality holds since  $X_{S^i}^* \cap X_T^*$  is a feasible replenishment policy for  $S$  which means that  $r_{S^i}^\Gamma(X_{S^i}^* \cap X_T^*) \geq r_S^\Gamma(X_S^*)$  therefore  $r_{S^i}^\Gamma(X_{S^i}^* \cap X_T^*) + r_i^w(E_i) \geq r_S^\Gamma(X_S^*) + r_i^w(E_i)$  and eventually  $r_{S^i}^\Gamma(X_{S^i}^* \cap X_T^*) \geq r_{S^i}^\Gamma(X_S^*)$ . Therefore,

$$r_{T^i}^\Gamma(X_{T^i}^*) - r_{T^i}^\Gamma(X_T^*) \leq r_{S^i}^\Gamma(X_{S^i}^*) - r_{S^i}^\Gamma(X_S^*).$$

By adding  $r_i^w(E_i)$  to both sides of the above inequality and noting (with the help of Lemma 1) that  $r_T^\Gamma(X_T^*) = r_{T^i}^\Gamma(X_T^*) - r_i^w(E_i)$  and  $r_S^\Gamma(X_S^*) = r_{S^i}^\Gamma(X_S^*) - r_i^w(E_i)$  we get

$$r_{T^i}^\Gamma(X_{T^i}^*) - r_T^\Gamma(X_T^*) \leq r_{S^i}^\Gamma(X_{S^i}^*) - r_S^\Gamma(X_S^*).$$

Thus we have  $c^\Gamma(T^i) - c^\Gamma(T) \leq c^\Gamma(S^i) - c^\Gamma(S)$ . □



In light of the sufficient condition for submodularity of CRI situation presented in Theorem 1, one can infer that the CRI games associated with situations with submodular cost components are concave.

## 6.2 Allocation Rules

An important question in every cooperative situation concerns the division of joint costs among the participants. The subadditivity of CRI games has already established that cooperation among players in their replenishment activities could reduce their total costs. However, since the decisions with regard to joining the collaborative organization are made by rational and self-interested players, it is crucially important to ensure that when dividing the total costs, every player is satisfied with its individual allocation. This is the main theme of the cost-sharing problem discussed in this section.

There are certain properties that a desirable allocation must satisfy. One of the most basic desirable properties of an allocation is the *efficiency property* which requires that the total cost of the set of all players (grand coalition) is entirely divided among the players. Another desirable property is the *stability property* that ensures players do not break apart from the grand coalition. The allocations in the core satisfy both of these properties. Due to the appealing features of allocations in the core, an important question with regard to every cooperative game is the nonemptiness of its core.

In general the core of CRI games can be empty. Below we provide an example where the core of the game associated with a CRI situation is empty.

**Example 1.** Consider the situation  $\Gamma$  as following. There are three players  $N = \{1, 2, 3\}$ , replenishing a single product  $E_1 = E_2 = E_3 = \{a\}$ . The cost of replenishment from the intermediaries are 4 for all players, i.e.,  $r_i^w(\{a\}) = 4$  for all  $i \in N$ . The cost of replenishment from the manufacturer is as follows:  $r_a^m(S) = 5$  if  $|S| = 1$ ,  $r_a^m(S) = 5$  if  $|S| = 2$ , and  $r_a^m(N) = 9$ . It is straightforward to check that in this situation we have  $c^\Gamma(S) = 4$  if  $|S| = 1$ ,  $c^\Gamma(S) = 5$  if  $|S| = 2$ , and  $c^\Gamma(N) = 9$ . The game is symmetric so if the core is not empty, then the equal allocation of  $9/3 = 3$  for every player must be in the core. However, every two player coalition can achieve the cost of 5 which is smaller than  $3 + 3 = 6$ . Thus the core of the game associated with  $\Gamma$  is empty.  $\triangle$

Nevertheless, the core of a concave game is always non-empty (Shapley, 1971). Our results in the previous section regarding the concavity of CRI games also guarantees the existence of allocations in the core.

**Corollary 1.** For every  $\Gamma \in \mathbf{\Gamma}^{sm}$ , the core of the associated cooperative CRI game  $(N, c^\Gamma)$  is non-empty.

When the core of a cooperative game is non-empty, it may contain an infinite number of distinct allocations. To provide insights about the nature of allocations in the core of CRI games, we present an observation with regard to the minimum amount of payment that every player must pay in core allocations.

**Theorem 4.** Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$  such that the core of its associated game  $(N, c^\Gamma)$  is non-empty. For every allocation  $\beta$  in the core, every optimal replenishment policy for the grand coalition  $X^*$ , and every player  $i \in N$  it holds that  $\beta_i \geq r_i^w(P_i^\Gamma[X^*])$ .

*Proof.* Let  $\beta$  be an allocation in the core of  $(N, c^\Gamma)$ . By definition of core allocations, it must be that  $\sum_{j \in N \setminus i} \beta_j \leq c^\Gamma(N \setminus i)$ . Also the efficiency of  $\beta$  requires that  $\sum_{j \in N} \beta_j = c^\Gamma(N)$ . Thus we can write

$$\beta_i = c^\Gamma(N) - \sum_{j \in N \setminus i} \beta_j \geq c^\Gamma(N) - c^\Gamma(N \setminus i)$$

Let  $X^*$  be an optimal replenishment policy for  $N$  and consider a player  $i \in N$ . To complete the proof it suffices to show that  $c^\Gamma(N) - c^\Gamma(N \setminus i) \geq r_i^w(P_i^\Gamma[X^*])$ . Let  $X = X^* \cap \mathcal{X}_{N \setminus i}^\Gamma$ . For every  $j \in N \setminus i$  it holds that  $P_j^\Gamma[X] = P_j^\Gamma[X^*]$ . Also we have  $I_l^\Gamma[X] = I_l^\Gamma[X^*] \setminus i$ . Since  $X$  is a feasible replenishment policy for  $N \setminus i$ , we can write:

$$r_{N \setminus i}^\Gamma(X) = \sum_{j \in N \setminus i} r_j^w(P_j^\Gamma[X^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X^*] \setminus i)$$

Every optimal replenishment policy for  $N \setminus i$  is at most as costly as  $X$ , thus  $c^\Gamma(N \setminus i) \leq r_{N \setminus i}^\Gamma(X)$ . We have

$$\begin{aligned} c^\Gamma(N) - c^\Gamma(N \setminus i) &\geq c^\Gamma(N) - r_{N \setminus i}^\Gamma(X) \\ &= \sum_{j \in N} r_j^w(P_j^\Gamma[X^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X^*]) - \sum_{j \in N \setminus i} r_j^w(P_j^\Gamma[X^*]) - \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X^*] \setminus i) \\ &= r_i^w(P_i^\Gamma[X^*]) + \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[X^*]) - r_l^m(I_l^\Gamma[X^*] \setminus i). \end{aligned}$$

For every  $l \in \mathcal{E}$  the function  $r_l^m$  is non-decreasing on  $N$  thus  $r_l^m(I_l^\Gamma[X^*]) - r_l^m(I_l^\Gamma[X^*] \setminus i) \geq 0$  and consequently we have  $c^\Gamma(N) - c^\Gamma(N \setminus i) \geq r_i^w(P_i^\Gamma[X^*])$  which completes the proof.  $\square$

Theorem 4 asserts that in every core allocation, each player has to pay at least its indirect replenishment cost for the products it obtains from the intermediaries. Therefore, irrespective of the contribution of a player to the total cost savings in the grand coalition, the indirect replenishment cost of no player would be subsidized in any core allocation.

In order for the cooperating organization to be able to repeatedly carry out joint replenishments without the need of renegotiating the appropriate allocations, a formal scheme for allocating the costs in different situations should be in place. This requirement is formalized with the notion of allocation rule. An *allocation rule* is a function  $\sigma$  which determines an allocation for every game in its domain of definition. The desirability of an allocation rule can be evaluated by the desirable properties of the allocations it generates. For example, an allocation rule is called *efficient* if it always generates efficient allocations. The allocation to player  $i$  under allocation rule  $\sigma$  is denoted with  $\sigma_i$ .

A well-known allocation rule in cooperative games literature is the celebrated *Shapley value* (Shapley, 1953). The Shapley value of a cost game<sup>2</sup>  $(N, c)$ , i.e.,  $\Phi(N, c)$ , is calculated

<sup>2</sup>Note that the Shapley value is originally proposed for saving games.

by the following formula:

$$\Phi_i(N, c) = \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} [c(S \cup i) - c(S)], \quad \text{for every } i \in N. \quad (7)$$

The Shapley value divides the total cost of grand coalition according to the average contributions of players in all subsets that they are a member of. Although in general the Shapley value of a game might not belong to its core, in concave games the latter is always the case (Shapley, 1971). Therefore, in submodular CRI games players can always divide the costs among themselves in a stable and efficient way by implementing the Shapley value. In the next section we demonstrate another appealing property of the Shapley value for cooperative CRI games.

## 7 Strategic Participation in Two-stage CRI Games

An implicit assumption made in the cooperative CRI game studied in previous section was that once a player decides to join the collaborative organization, it puts forward its entire player-specific product set so that their replenishment sources are decided by the collaborative organization in order to optimize the replenishment cost of the grand coalition. However, in reality the players' decisions with regard to their participation in collaborative replenishment activities is more nuanced. One of the most important dimensions of such decisions is the extent of the players' participation in the collaborative organization in terms of the products whose replenishment policies are delegated to the collaborative organization. In this section, the players have the option to strategically choose the products they replenish individually outside of the collaborative organization. Thus, by allowing players to withhold some of their required products from the collaborative organization, they are able to *partially* collaborate. The participation decision with respect to each product is binary, i.e., each product is entirely replenished either within the collaborative organization or outside it. The question we investigate is the conditions under which centrally optimal outcomes would be achieved decentrally, i.e., strategic participation of the players would not negatively affect the total replenishment cost of the supply chain.

A crucial input to the players' strategic decision making processes is the allocation rule that will be implemented in the collaborative organization to divide the joint costs. Hence, our analysis in this section enables us to comment on appropriate allocation rules for collaborative organizations in CRI situations. In order to achieve the latter, we construct a two-stage game comprising a non-cooperative stage followed by a subsequent cooperative stage. The sequence of events in our two-stage CRI game is as follows. First, the allocation rule for the collaborative organization is set and announced to the players. With this knowledge the players simultaneously make their decisions regarding the extend of their participation in the collaborative organization. That is, each player strategically chooses the products it would replenish by itself outside the collaborative organization from the intermediaries and announces the rest of its products to the collaborative organization. The cooperative CRI game played in the second stage is associated with the modified version of

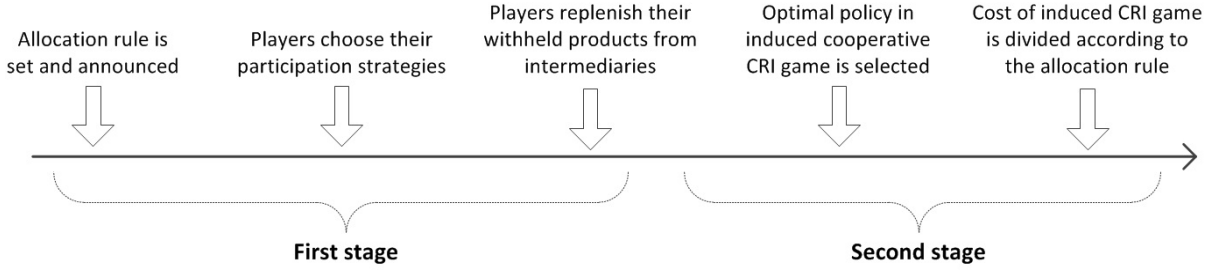


Figure 2: Sequence of events in a two-stage CRI game

the original CRI situation which is induced by the players' participation strategies in the first stage. The total cost of the grand coalition in the induced CRI situation in the second stage will be distributed according to the pre-fixed allocation rule. Figure 2 illustrates the sequence of events. We start by assuming that all information contained in the situation is known by all players.

We formulate the players' participation strategies in two-stage CRI games in terms of products they withhold from the collaborative organization and replenish individually from the intermediaries (this formulation of strategies is for notational convenience). Given a CRI situation  $\Gamma = (N, E, r^w, r^m)$  and a player  $i \in N$ , let  $L_i \subseteq E_i$  be the set of withheld products of  $i$ . In this manner,  $L_i$  is the participation strategy of player  $i$ . A vector of players' strategies  $L = (L_i)_{i \in N}$  is referred to as a *participation strategy profile*.

As players withhold some of their products from the collaborative organization, the cooperative game they play in the second stage can be associated with a situation different than the original CRI situation. A participation strategy profile induces a CRI situation wherein only the products intended by the players are present. The modified situation induced by the participation strategy profile  $L$  is denoted by  $\Gamma \setminus L$  and obtained in the following manner:

$$\Gamma \setminus L = (N, E \setminus L, r^w, r^m) \quad (8)$$

where  $E \setminus L = (E_i \setminus L_i)_{i \in N}$  is the modified vector of player-specific product sets. Subsequently, the game associated with the modified situation, to be played in the second stage, is defined by  $(N, c^{\Gamma \setminus L})$ . The following lemma present a useful observation regarding the modified CRI situations.

**Lemma 3.** *Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$ , and  $L = (L_i)_{i \in N}$  with  $L_i \subseteq E_i$  for all  $i \in N$ . For every  $S \subseteq T \subseteq N$  we have,*

$$c^{\Gamma \setminus L - S}(T) \leq \sum_{i \in S} r_i^w(L_i) + c^{\Gamma \setminus L}(T). \quad (9)$$

*Proof.* Let  $X^*$  be an optimal replenishment policy for  $T$  in situation  $\Gamma \setminus L$ . The left hand

side of inequality (9) can be written as

$$\begin{aligned}
\sum_{i \in S} r_i^w(L_i) + c^{\Gamma \setminus L}(T) &= \sum_{i \in S} r_i^w(L_i) + r_T^{\Gamma \setminus L}(X^*) \\
&= \sum_{i \in S} r_i^w(L_i) + \sum_{i \in T} r_i^w(P_i^{\Gamma \setminus L}[X^*]) + \sum_{l \in \mathcal{E}} (I_l^{\Gamma \setminus L}[X^*]) \\
&\geq \sum_{i \in T} r_i^w(P_i^{\Gamma \setminus L-S}[X^*]) + \sum_{l \in \mathcal{E}} (I_l^{\Gamma \setminus L-S}[X^*]) \\
&= r_T^{\Gamma \setminus L-S}(X^*) \\
&\geq c^{\Gamma \setminus L-S}(T).
\end{aligned}$$

The first inequality uses that subadditive property of  $r_i^w$  and the fact that  $I_l^{\Gamma \setminus L}([X^*]) = I_l^{\Gamma \setminus L-S}([X^*])$ . The last inequality follows since  $X^*$  is also a feasible replenishment policy for  $T$  in situation  $\Gamma \setminus L-S$  and consequently its corresponding cost is never less than an optimal policy for  $T$  in that situation.  $\square$

Lemma 3 states that the replenishment cost of a group of players never decreases if a subgroup of players withhold some of their products from the collaborative organization and replenish them individually from the intermediaries. This observation is justified by the fact that if it is to the benefit of the group that certain players replenish parts of their product sets from the intermediaries, the group's optimal replenishment policy would recommend this.

Given a CRI situation  $\Gamma = (N, E, r^w, r^m)$  and an allocation rule  $\sigma$  for CRI games, the two-stage participation game under allocation rule  $\sigma$  is the triple  $(N, \wp(E), z^{\Gamma, \sigma})$  where  $\wp(E) = (\wp(E_i))_{i \in N}$  is the vector of individual participation strategy sets—i.e., power sets of player-specific product sets—and  $z^{\Gamma, \sigma}$  is the vector of player-specific cost functions with its  $i$ 'th element,  $i \in N$ , defined such that for a participation strategy profile  $L$  we have

$$z_i^{\Gamma, \sigma}(L) = r_i^w(L_i) + \sigma_i(N, c^{\Gamma \setminus L}). \quad (10)$$

The player-specific cost function of player  $i$  is comprised of the indirect replenishment cost of player  $i$  for its withheld products and its allocation under  $\sigma$  in the cooperative CRI game induced by  $L$ , i.e., the game associated with the modified CRI situation  $\Gamma \setminus L$ .

In two-stage CRI games, the individual decision making processes of the players are intertwined as the player-specific cost functions of players will be affected by the other players' choices of strategies as well. The rational players choose their individual participation strategies in anticipation of the other players' moves in order to minimize their player-specific cost functions. A particularly interesting outcome for the system is when the strategic choices of players coincide with the strategies that minimize the total replenishment costs of the entire system, i.e., when a *centrally optimal participation strategy profile* is selected individually by the players. In the latter case the supply chain would be coordinated. The following lemma highlights a centrally optimal participation strategy profile in two-stage CRI games.

**Lemma 4.** *Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$  be a CRI situation and  $\sigma$  be an efficient allocation rule for cooperative CRI games. In the two-stage CRI game associated with  $\Gamma$ , the complete*

participation strategy profile  $L^\circ = (L_i^\circ = \emptyset)_{i \in N}$  minimizes the sum of player-specific cost functions so that the total equals  $c^\Gamma(N)$ .

*Proof.* For a strategy profile  $L \in \prod_{i \in N} \wp(E_i)$  the sum of player-specific cost functions in (10) can be written as

$$\begin{aligned} \sum_{i \in N} z_i^{\Gamma, \sigma}(L) &= \sum_{i \in N} [r_i^w(L_i) + \sigma_i(\Gamma \setminus L)] \\ &= \sum_{i \in N} r_i^w(L_i) + c^{\Gamma \setminus L}(N) \\ &\geq c^\Gamma(N) \end{aligned}$$

where the second equality follows from efficiency of  $\sigma$  and the inequality is deduced from Lemma 3. Therefore, the sum of player-specific cost functions of all players with any participation strategy profile cannot be less than  $c^\Gamma(N)$ . For complete participation strategy profile  $L^\circ$  we have  $\sum_{i \in N} z_i^{\Gamma, \sigma}(L^\circ) = c^\Gamma(N)$ . Hence,  $L^\circ$  is an optimal participation strategy for  $N$ .  $\square$

With the choice of complete participation strategy profiles there would be no loss of efficiency in the two-stage CRI games. However, despite the central optimality of complete participation strategy profiles in two-stage CRI games, players may choose other participation strategies if such strategies result in lower player-specific costs for them. A critical variable in this setting is the allocation rule chosen for cooperative games played in the second stage. The following definition captures the formal relation between the choice of allocation rules and the players' participation strategies in two-stage CRI games.

**Definition 2.** *The allocation rule  $\sigma$  implements the participation strategy profile  $L$  in Nash equilibrium (or weakly dominant strategies) in a two-stage CRI game if  $L$  is a Nash equilibrium (or weakly dominant strategy profile) in that game.*

Definition 2 introduces two types of implementations. Remember from Section 3 that every weakly dominant strategy profile is also a Nash equilibrium. Thus, if an allocation rule could implement a participation strategy profile in weakly dominant strategies it can also implement that strategy in Nash equilibrium. The reverse, however, may not hold necessarily. It has been argued that implementation in (weakly) dominant strategies is the most demanding form of implementation (Maskin and Sjöström, 2002). In the next step we present the main result of this section regarding the ability of the Shapley value to implement centrally optimal participation strategy profiles in two-stage CRI games.

**Theorem 5.** *The Shapley value implements the complete participation strategy profile in weakly dominant strategies in every two-stage CRI game.*

*Proof.* Let  $\Gamma = (N, E, r^w, r^m) \in \mathbf{\Gamma}$  and  $(N, \wp(E), z^{\Gamma, \sigma})$  be its associated two-stage CRI game. We show that for every player  $i \in N$  and every  $L \in \prod_{i \in N} \wp(E_i)$ , it holds that  $z_i^{\Gamma, \Phi}(L) \geq z_i^{\Gamma, \Phi}(L_{-i}, \emptyset)$ , or equivalently

$$r_i^w(L_i) + \Phi_i(N, c^{\Gamma \setminus L}) \geq \Phi_i(N, c^{\Gamma \setminus L_{-i}}). \quad (11)$$

Using the definition of the Shapley value in (7), the left hand side of (11) can be written as

$$\begin{aligned}
r_i^w(L_i) + \Phi_i(N, c^{\Gamma \setminus L}) &= r_i^w(L_i) + \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} [c^{\Gamma \setminus L}(S^i) - c^{\Gamma \setminus L}(S)] \\
&= \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} [r_i^w(L_i) + c^{\Gamma \setminus L}(S^i) - c^{\Gamma \setminus L-i}(S)] \\
&\geq \sum_{S \subseteq N \setminus i} \frac{|S|!(n - |S| - 1)!}{n!} [c^{\Gamma \setminus L-i}(S^i) - c^{\Gamma \setminus L-i}(S)] \\
&= \Phi_i(N, c^{\Gamma \setminus L-i}).
\end{aligned}$$

The second equality uses the facts that  $\sum_{S \subseteq N \setminus i} |S|!(n - |S| - 1)!/n! = 1$  and  $c^{\Gamma \setminus L}(S) = c^{\Gamma \setminus L-i}(S)$  (since  $i \notin S$ ). The inequality follows from Lemma 3.  $\square$

Theorem 5 exhibits an appealing feature of the Shapley value in CRI situations. That is, if the Shapley value is set as the allocation rule, no player can obtain any benefit by withholding some of its products from the collaborative organization. The power of the Shapley value in enforcing the centrally optimal strategies in CRI situations becomes clearer once we realize that the complete participation strategy is a feasible choice for every player in every CRI situation. The next observation follows immediately.

**Corollary 2.** *Let  $i \in N$  be a player. With the Shapley value as the allocation rule, the complete participation strategy  $L_i^o = \emptyset$  is an optimal strategy for player  $i$  in every two-stage CRI game.*

Corollary 2 has important consequences in terms of the information available to every player and its effect on the choice of centrally optimal participation strategies. Since complete participation strategies are always best choices of strategies at the individual level under the Shapley value, the players do not need to know the specific details of the situation in order to realize that announcing their complete player-specific product sets to the collaborative organization is their best options. We conclude that the Shapley value can lead to the coordination of the decentralized system under study even in settings without complete information.

## 8 Disregarding Intermediaries in the Cooperative Stage

Drawing upon the logic of two-stage games elaborated upon previously, the purpose of this section is to answer the following question: is it really necessary to consider the replenishment options from the intermediaries in the cooperative stage? Alternatively, what happens if the collaborative organization disregards the options to replenish from intermediaries in the second stage? The motivation for this question is two-fold. First, it has already been established that in every optimal replenishment policy, one can separate the player-product pairs that are replenished from the intermediaries from those that are replenished from the

manufacturers. Thus, in a strategic game where players are free to withhold some of their products and replenish them from the intermediaries, there can be strategy profiles with partial cooperation which result in minimum total costs of corresponding centralized system. Second, from practical point of view it might be easier to set up the collaborative organization to only deal with direct replenishments from the manufacturers and replenishments from intermediaries be left out for players to manage individually. In this section we prove that for the optimal performance of the system it is vital that indirect replenishment options also be considered in the collaborative organization.

To carry out the analysis, we construct an alternative cooperative game which disregards the options to replenish from the intermediaries. Given the CRI situation  $\Gamma = (N, E, r^w, r^m)$ , define the *direct CRI game*  $(N, \tilde{c}^\Gamma)$  where for every  $S \subseteq N$ :

$$\tilde{c}^\Gamma(S) = r_S^\Gamma(\mathcal{X}_S^\Gamma) = \sum_{l \in \mathcal{E}} r_l^m(I_l^\Gamma[\mathcal{X}_S^\Gamma]) \quad (12)$$

The cost to every coalition in the direct CRI game is the direct replenishment cost of all products of every player. In this manner, direct CRI games disregard the intermediaries. Subsequently, we define an alternative two-stage game associated with CRI situations in the same spirit as in the previous section. In the first stage of this alternative two-stage game, each player decides its withheld product set which would be replenished individually from the intermediaries. In the second stage of the alternative two-stage game, the direct CRI game induced by the chosen strategies is played and the costs will be divided according to a pre-fixed allocation rule for the direct CRI games. We refer to these two-stage games as the *alternative two-stage CRI games*.

The alternative two-stage CRI game associated with situation  $\Gamma = (N, E, r^w, r^m)$  under allocation rule  $\sigma$  is the triple  $(N, \wp(E), \tilde{z}^{\Gamma, \sigma})$  where  $\wp(E) = (\wp(E_i))_{i \in N}$  is the vector of individual participation strategy sets and  $\tilde{z}^{\Gamma, \sigma}$  is the vector of alternative player-specific cost functions with its  $i$ 'th element,  $i \in N$ , defined such that for a strategy profile  $L \in \prod_{i \in N} \wp(E_i)$  we have

$$\tilde{z}_i^{\Gamma, \sigma}(L) = r_i^w(L_i) + \sigma_i(N, \tilde{c}^{\Gamma \setminus L}). \quad (13)$$

From the above definition it must be evident that there is a bijection between the set of participation strategies for an alternative two-stage CRI game and the replenishment choice set of players in the situation. Drawing upon the latter fact, the next observation, which we provide without proof, shows a centrally optimal participation strategy profile for an alternative two-stage CRI game that minimizes the sum of player-specific cost functions of the players so that the total boils down to the minimum total replenishment cost of the corresponding centralized system.

**Lemma 5.** *Let  $\Gamma = (N, E, r^w, r^m)$  be a CRI situation,  $X^*$  be an optimal replenishment policy for  $N$ , and  $\sigma$  be an efficient allocation rule for direct CRI games. In the alternative two-stage CRI game associated with  $\Gamma$ , the participation strategy profile  $L^P = P^\Gamma[X^*]$  minimizes the sum of player-specific cost functions so that the total equals  $c^\Gamma(N)$ .*

Lemma 5 asserts that if the supply chain is managed centrally, then there would be no efficiency lost at optimality when the options to replenish from intermediaries are disregarded



in the second stage. In fact, minimum total replenishment cost of the centralized system can be achieved if each player withholds the same set of products from the collaborative organization that it would have replenished from intermediaries in an optimal replenishment policy for the corresponding CRI situation. But is there an allocation rule for direct CRI games that motivate players to choose latter participation strategy profiles in alternative two-stage CRI games? Modifying the notion of implementation in Definition 2 for alternative two-stage CRI games, next we investigate the existence of allocation rules that could implement centrally optimal participation strategy profiles in these games. As we show below, such allocation rules do not exist even if we require implementation in the weaker form, i.e., in Nash equilibrium.

**Theorem 6.** *There exists no efficient allocation rule for direct CRI games that could implement the centrally optimal participation strategy profiles in every alternative two-stage CRI game.*

*Proof.* Consider the CRI situation  $\Gamma$  with  $N = \{1, 2\}$ ,  $E_1 = \{a, b\}$  and  $E_2 = \{a\}$ . For  $i \in N$  we have  $r_i^w(\{a\}) = r_i^w(\{b\}) = 9$  and  $r_i^w(\{a, b\}) = 16$ , and for  $l \in \mathcal{E}$  we have  $r_l^m(\{1\}) = r_l^m(\{2\}) = 10$  and  $r_l^m(N) = 15$ . It can be seen that  $c^\Gamma(\{1\}) = r_1^w(E_1) = 16$ ,  $c^\Gamma(\{2\}) = r_2^w(E_2) = 9$ , and  $c^\Gamma(N) = r_1^w(\{b\}) + r_2^m(N) = 9 + 15 = 24$ . Observe that the only centrally optimal participation strategy profile in the alternative two-stage game associated with  $\Gamma$  is  $L^P = (L_1^P = \{b\}, L_2^P = \emptyset)$ . The modified situation associated with the latter participation strategy profile is  $\Gamma \setminus L^P = (\{1, 2\}, (\{a\}, \{a\}), r^w, r^m)$ .

Next, consider the situation  $\hat{\Gamma}$  with  $N = \{1, 2\}$ ,  $E_1 = \{a\}$  and  $E_2 = \{a, b\}$  and cost components that are identical to those in  $\Gamma$ . It can be seen that  $c^{\hat{\Gamma}}(\{1\}) = 9$ ,  $c^{\hat{\Gamma}}(\{2\}) = 16$ ,  $c^{\hat{\Gamma}}(N) = 24$ . The unique optimal participation strategy profile in the alternative two-stage game associated with  $\hat{\Gamma}$  is  $\hat{L}^P = (\hat{L}_1^P = \{b\}, \hat{L}_2^P = \emptyset)$ . The modified situation associated with the latter participation strategy profile is  $\hat{\Gamma} \setminus \hat{L}^P = (\{1, 2\}, (\{a\}, \{a\}), r^w, r^m)$ . As seen above, the modified situations associated with  $\Gamma$  and  $\hat{\Gamma}$  are identical thus cooperative games associated with them are also identical. For the direct CRI game associated with situation  $\dot{\Gamma} = \Gamma \setminus L^P = \hat{\Gamma} \setminus \hat{L}^P$  we have  $\tilde{c}^{\dot{\Gamma}}(\{1\}) = \tilde{c}^{\dot{\Gamma}}(\{2\}) = 9$ , and  $\tilde{c}^{\dot{\Gamma}}(N) = 15$ .

Suppose that an efficient allocation rule  $\sigma$  is chosen that divides the costs between the players in such a way that  $\sigma_1(N, \tilde{c}^{\dot{\Gamma}}) \geq \sigma_2(N, \tilde{c}^{\dot{\Gamma}})$ . Consider the situation  $\Gamma$ . With allocation rule  $\sigma$ , it would be the case that

$$\tilde{z}_1^{\Gamma, \sigma}(L^P) = r_1^w(\{b\}) + \sigma_1(N, \tilde{c}^{\Gamma \setminus L^P}) \geq 9 + 7.5 = 16.5.$$

In this situation, if player 1 deviates from choosing  $L_1^P$  and instead chooses  $L_1 = \{a, b\}$ , while player 2 chooses  $L_2^P$ , player 1 would get  $\tilde{z}_1^{\Gamma, \sigma}(L_1, L_2^P) = 16 < \tilde{z}_1^{\Gamma, \sigma}(L^P)$ . Therefore the allocation rule which gives player 1 a cost allocation which is equal to or higher than that of player 2 cannot implement the optimal participation strategy profile in Nash equilibrium in this situation.

To remedy this, suppose a different allocation rule  $\sigma'$  is chosen such that  $\sigma'_1(N, \tilde{c}^{\dot{\Gamma}}) < \sigma'_2(N, \tilde{c}^{\dot{\Gamma}})$ . However, once we consider the original situation  $\hat{\Gamma}$ , it can be seen that an allocation rule that gives player 2 a higher allocation than 1 is unable to implement the corresponding centrally optimal participation strategy profile in Nash equilibrium. We conclude

that there exists no allocation rule that could implement the optimal participation strategy profiles in Nash equilibrium in alternative two-stage CRI games associated with situations  $\Gamma$  and  $\hat{\Gamma}$  simultaneously.  $\square$

In light of last result we conclude that the optimal participation of players in collaborative replenishment organizations which disregard the presence of intermediaries cannot be guaranteed. Hence, to obtain the system wide optimal performance, it is crucial that the collaborative replenishment organizations do consider the players' options for replenishments from the intermediaries.

## 9 Final Remarks

In this paper, we investigated potential opportunities for direct replenishments from manufacturers for collaborating downstream buyers when supply chain intermediaries provide an alternative source for replenishments. In a typical situation with the intermediaries offering low order costs, possibility to bundle multiple products in one order, yet higher unit costs than manufacturers, the incentives for replenishing from different sources are conflicting. The main insight obtained from our study is that under certain conditions cooperation enables downstream buyers to bypass the intermediaries and directly replenish from the original manufacturers. However, the indirect replenishment from the intermediaries may still be a part of the optimal replenishment policy. For example, the optimal replenishment policy might require buyers to obtain low-demand products from the intermediaries and high-demand products from the manufacturers. When possible, the downstream cooperation increases the supply chain efficiency by eliminating double marginalization and excessive inventories.

We showed that the Shapely value possesses several desirable properties for being the allocation rule of choice in CRI situations. Firstly, it would provide stable allocations for a considerable category of CRI situations which extend the existing submodular single-source joint replenishment models in the literature. Secondly, the Shapley value implements complete participation strategy profiles in all CRI situations in such a way that for every player, delegating the replenishment decisions of all products to the collaborative organization is the best strategy, even if no information about the other players is known. We further showed that if replenishments from the intermediaries are viable options in the supply chain, disregarding them in the collaborative organization hinders the coordination of the supply chain.

There are many other perspectives to consider when horizontal collaboration in supply chains are carried out in the presence of intermediaries. Joint replenishment activities are likely to affect the pricing schemes of manufacturers, intermediaries, and downstream buyers. So an important direction for future research is to study the dynamics stemming from price competitions among supply chain entities. This would be in line with the work of James and Dana (2012) who study the impact of collaborative purchasing organizations on price competition among the suppliers. It is worth mentioning that although collaborative purchasing results in lower purchasing prices for the downstream players, competition in price-setting may leave them worse off—an instance of this situation discussed by Chen and

Roma (2011). Another possible extension of our work is to address the additional costs faced by the organizations of collaborating buyers. It has been observed in the literature, e.g., in Hezarkhani and Kubiak (2013), that increasing collaboration costs can be a threat to the stability in supply chains. Hence, it is important to understand to what extent collaborative organizations can afford the increasing costs of required communication, negotiations, and infrastructure. We leave these for future research.

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